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PROGRESSIVE FAILURE OF ADVANCED COMPOSITE LAMINATES USING THE F--ETC(U)
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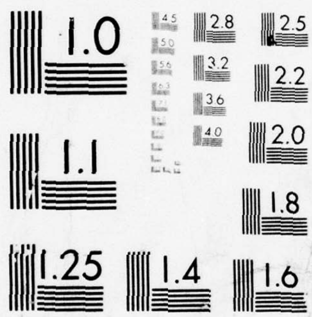

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PROGRESSIVE FAILURE
OF ADVANCED COMPOSITE LAMINATES
USING THE FINITE ELEMENT METHOD

By

Gary Earl Brown

A thesis submitted to the faculty of the
University of Utah in partial fulfillment of the requirements
for the degree of

Master of Science

Department of Mechanical Engineering

University of Utah

March 1976

UNIVERSITY OF UTAH GRADUATE SCHOOL

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In addition, the patience, understanding and encouragement of my wife, Mary, and my children, merits a special note of appreciation.

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LIST OF FIGURES

NOMENCLATURE

[A]	= system stiffness matrix
a_i	= coefficient in area coordinate equation
$[B_{ij}]$	= laminate coupling stiffness matrix
[B]	= strain-displacement matrix
b_i	= coefficient in area coordinate equation
c_i	= coefficient in area coordinate equation
$[D_{ij}]$	= laminate inplane stiffness matrix
E_{11}	= Young's modulus in the 1 direction
E_{22}	= Young's modulus in the 2 direction
e	= superscript indicates element value
{F}	= nodal force array
G_{12}	= lamina shear modulus
i,j,k,m,n	= dummy indices
[K]	= element stiffness matrix
M_x, M_y, M_{xy}	= moment resultants
N_x, N_y, N_{xy}	= stress resultants
{PF}	= pseudo nodal force vector
PF_{max}	= maximum pseudo nodal force
[Q]	= laminate stiffness matrix in material coordinate system
$[\bar{Q}]$	= laminate stiffness matrix in arbitrary coordinate system
S_ε	= maximum allowable shear strain

$[T]$ = transformation matrix

t = laminate thickness

U_{1-5} = invariant material constants for lamina

U_i = nodal force, x-direction

U = strain energy

u = displacement, x-direction

V_i = nodal force, y-direction

v = displacement, y-direction

W = work

w = displacement, z-direction

$X_{\varepsilon t}$ = maximum allowable tensile strain, 1 direction

$X_{\varepsilon c}$ = maximum allowable compressive strain, 1 direction

x,y,z = arbitrary coordinate system

$Y_{\varepsilon t}$ = maximum allowable tensile strain, 2 direction

$Y_{\varepsilon c}$ = maximum allowable compressive strain, 2 direction

α_i = constant

Δ = indicates change in variable

γ = shear strain

$\{\delta\}$ = displacement vector

ε = normal strain

θ = angular displacement

κ = curvature

σ = normal stress

τ = shear stress

ν_{12} = major Poisson's ratio

ν_{21} = minor Poisson's ratio

Δ = area, triangular element

ABSTRACT

In the study of fiber-reinforced resin composites, the analysis of the progressive failure of a laminate with a stress concentration subjected to plane stress poses a very interesting but complex problem. This thesis approaches this problem by using the finite element method to examine the progressive failure of symmetrical laminates.

A modified maximum strain failure theory is proposed and a finite element computer program developed that accounts for progressive failure. A computer analysis of several unnotched laminate tensile specimens, with lamina at various angles, was made and these results are compared with experimental data. degree

degree Circular hole tensile specimens with $(0°/90°/90°/0°)_s$ and $(0°/\pm 45°/90°)_s$ lamina were also investigated, and the progressive failure through the finite element grid presented. The ultimate failure loads of the circular hole specimens are compared with experimental data. Material properties used were those for Thornel 300/5208 Graphite-Epoxy.

Although the results obtained cannot be considered conclusive for all cases, they do compare favorably with experimental data for the unnotched specimens. The ultimate failure loads of the hole specimens were somewhat higher than those obtained experimentally.

INTRODUCTION

The word "composite" in composite material signifies that two or more materials are combined on a macroscopic scale to form a useful material. The advantage of composites is that the materials can be combined in ways that usually exhibit the best qualities of their constituents and often some qualities that neither constituent possesses. Some properties that may be improved by use of a composite material are strength, stiffness, weight, corrosion resistance, fatigue life, and thermal properties.[1]

Composite materials have been used for centuries. When the first composite was used is unknown, but recorded history contains references to various forms of composite materials. For example, the Egyptians used laminated wood as early as 2780 B.C., and the Israelites added chopped straw to the manufacture of bricks in 800 B.C.[2] A short time thereafter, the Mongol bow was developed from a composite of animal tendons, wood and silk bonded together with an adhesive. Still later, laminated structures appeared in the Damascus gun barrels and Japanese ceremonial swords.[3]

More recently, fiber-reinforced resin composites that have a high strength-to-weight and stiffness-to-weight ratio have become important in weight sensitive applications such as aircraft and space vehicles. Some examples of these modern applications of fiber-resin composites are: an AT-6C aircraft with reinforced plastic fuselage

built in 1943,[4] helicopter rotor blades specifically designed to reduce vibration and withstand torsional loads,[4,5] rocket motor cases,[1] space vehicle structural components,[7] and presently an increasing number of uses such as fuselage and stabilizer components on the F-111, F-14, F-15, and F-16 aircraft.[1,6] Through the use of structural compounds made of composite materials, strength-to-weight ratios have been increased 100 percent over that of comparable metal structures.[5] The impact of composites on jet engine performance may be even more dramatic, where an 800 percent increase in the thrust-to-weight index appears possible.[1]

The superior strength-to-weight ratio of these fibrous composites is related to the failure mechanisms of homogeneous materials where, generally, the actual strength is considerably lower than the theoretical atomic strength. The reason for this strength difference is the formulation and movement of dislocations in the homogeneous material. By forming a material into thin whiskers or fibers with a small cross section, conformity in the microstructure is enhanced, the probability of internal flaws is reduced, and the formation and movement of dislocations restricted, making it possible for the fiber or whisker to approach its theoretical strength.[8] Composite sheets or "lamina" with high longitudinal strength are formed by imbedding many of these high strength fibers longitudinally in a suitable matrix material. A composite "laminate" with the desired strength and stiffness properties may then be formed by combining layers of lamina together at various orientations.

Investigation of lamina strength has generally been approached from both micromechanical and macromechanical levels.[1,9,10] The

micromechanical approach, which treats a composite material as a heterogeneous continuum, has been used for simple lamina models.[13,31,32] Although theoretically justifiable, this approach has a major limitation in that the analysis required is extremely complex, and therefore, limited to very simple geometries.[31]

In general, macromechanical prediction of lamina failure has been approached from one of the following three theories: the maximum strain theory, maximum stress theory, or maximum work theory.[12,14] Of these three theories, the maximum work approach has been proven to be the most accurate when compared with experimental data.[1,12,13] However, this theory does not easily lend itself to the analysis of progressive failure because the damage to the composite cannot be described and put into post-failure relations. The maximum strain and maximum stress theories are well suited for a progressive failure type analysis, but the accuracy of these theories deteriorates when the fibers are at an angle between 15 and 60 degrees to an applied uniaxial load.[1] The reason for this loss of accuracy at intermediate angles is probably due to not considering the interactive effect on failure of combined shear and tension.

Failure of unnotched laminates has been approached by combining plies through lamination theory and applying a lamina failure theory to each ply. In doing so, the disadvantages of ply failure theories are carried through to the laminate. In addition, failure of one lamina as a failure criterion for the laminate is usually too conservative. Maximum work or distortional energy applications to laminate failure, such as that by Tasi-Wu,[3] have given good results for individual laminates, but new properties must be obtained for each

new laminate. Sendeckyj,[29] using the method of Sandhu,[36] successfully used lamina stress-strain data to predict the nonlinear response of angle and multi-ply laminates in uniaxial tension. The failure predictions were less successful, however. The success of this approach to progressive failure of notched laminates is still to be determined.

Because of the complexity of the micromechanical approach, macromechanical principles are usually employed to determine laminate behavior in the presence of notches. The anisotropy of the laminate makes the failure properties due to stress concentrations of particular interest in that the stress concentration factor can be considerably higher for composites than for isotropic materials. In addition, a hole and crack size effect has been observed on laminate strength. Macromechanical studies of notched laminate failure have used models such as the 'inherent flaw model' for holes by Waddoups[35] and an 'average' or 'point' stress approach for holes and cracks by Nuismer and Whitney.[30] However, both studies neglect the load-path dependent damage or progressive failure of the laminate. In doing so, none can be expected to have general applicability, especially when biaxial loading is considered.

The purpose of this thesis is to study progressive failure of notched laminates subjected to in plane loads.

This thesis:

(1) presents a modified maximum strain theory for individual plies and develops the post failure constitutive relations for the ply;

(2) develops an incremental finite element program which uses

laminated plate theory to account for stiffness changes in the laminate due to failure in the plies;

(3) investigates the stress-strain behavior to failure of several unnotched laminates (under uniaxial tension loads) and compares the results with experimental data;

(4) traces the progressive failure of two laminates containing circular holes and compares predicted failure loads with experimental data.

COMPOSITE FAILURE

Anisotropic Elasticity

With the advent and increased usage of fibers such as graphite in composite materials, the assumption that the material is isotropic is no longer valid. Graphite fibers are highly anisotropic, with the longitudinal stiffness being an order of magnitude greater than the transverse stiffness. Then, in order to analyze such fiber-reinforced composites, anisotropic elasticity must be employed.

For a three dimensional stress state, the generalized Hooke's Law for an anisotropic material is given by:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = [Q_{ij}] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad (1)$$

where the stiffness matrix Q_{ij} is symmetrical with 21 independent constants.[1]

If there are two orthogonal planes of material property symmetry, such as parallel and perpendicular to the fibers in a unidirectional fiber composite, symmetry will also exist relative to a third mutually orthogonal plane, and the material is said to be orthotropic. The stress-strain relations for an orthotropic material in a coordinate

system aligned with the principal material directions, or parallel and perpendicular to the fiber direction, becomes:[1]

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad (2)$$

where the stiffness matrix has been reduced to nine independent constants.

Lamina Constitutive Relationships

One pertinent assumption in establishing the constitutive or *stress-strain relationships for the lamina* of a laminated composite is that a lamina, when in a composite, is in a state of plane stress. For a state of plane stress, and with the lamina in the 1-2 plane as shown in Figure 1, the following stresses are assumed zero:

$$\sigma_3 = 0, \quad \tau_{23} = 0, \quad \tau_{13} = 0 \qquad (3)$$

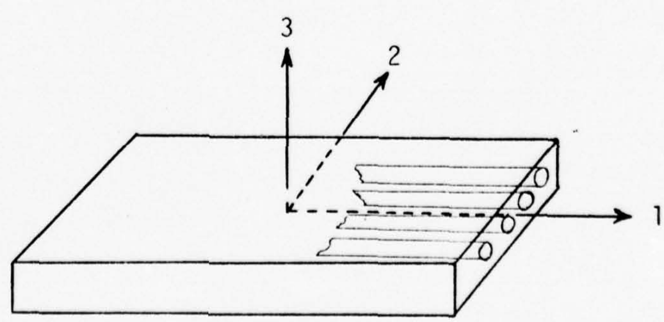


Figure 1. Unidirectional Lamina

By sutstituting into Equation (2), the stress-strain relation becomes:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{4}$$

where the components of the stiffness matrix for the orthotropic lamina given in terms of engineering constants are:

$$\begin{aligned} Q_{11} &= E_{11}/(1-\nu_{12}\ \nu_{21}) \\ Q_{22} &= E_{22}/(1-\nu_{12}\ \nu_{21}) \\ Q_{12} &= \nu_{21}E_{11}/(1-\nu_{12}\ \nu_{21}) = \nu_{12}E_{22}/(1-\nu_{12}\ \nu_{21}) \\ Q_{66} &= G_{12} \end{aligned} \tag{5}$$

There are now four independent constants: E_{11}, E_{22}, G_{12} and ν_{12}, which are the elastic moduli in the 1 and 2 directions, the shear modulus and the major Poisson's ratio, respectively. The major and minor Poisson's ratios are related by:

$$\nu_{21}E_{11} = \nu_{12}E_{22} \tag{6}$$

where the major Poisson's ratio, ν_{12}, is the ratio of strain in the 2 direction to strain in the 1 direction for a load in the one direction. To tailor a material with the proper stiffness and strength in various directions, unidirectional laminae are usually put together with fibers running in several different directions so that the lamina principal axes are not coincident with the reference axes of the laminate. When this occurs, the constitutive relations for each lamina must be transformed to the laminate reference axes in order to

determine the laminate constitutive relationship. The transformation relationship for stress between an arbitrary x-y axes and the primary 1-2 axes, as shown in Figure 2, is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{7}$$

while that for strain is

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} \tag{8}$$

where the transformation matrix T_{ij} is:

$$[T_{ij}] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \tag{9}$$

$$m = \cos\theta$$

$$n = \sin\theta$$

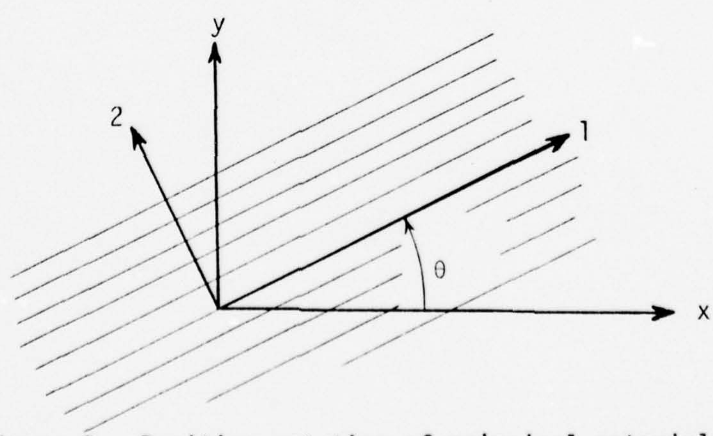


Figure 2. Positive rotation of principal material axes from arbitrary xy axes

In the same way, the primary stress and strain relations are referenced to the xy axes by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T_{ij}]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} = [T_{ij}]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \tag{11}$$

where T_{ij} inverse is obtained by substituting a negative angle θ for the positive angle θ in the T matrix. Thus, T inverse becomes:

$$[T_{ij}]^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix} \tag{12}$$

$$m = \cos\theta$$

$$n = \cos\theta$$

Knowing the orthotropic lamina material properties and referencing them to the xy axes, θ is measured in the negative direction. Then T becomes T^{-1} and the xy stress-strain relationship obtained from Equations (4), (8), and (10) is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T_{ij}]\,[Q_{ij}]\,[T_{ij}]^{-1} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{13}$$

where Q_{ij} is the orthotropic lamina stiffness from Equation (4).

Denoting the lamina stiffness with respect to the xy axes as $\bar{Q}_{ij}$, then

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}_{ij}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{14}$$

where

$$[\bar{Q}_{ij}] = [T_{ij}]\,[Q_{ij}]\,[T_{ij}]^{-1} \tag{15}$$

Upon multiplication, the terms of the $\bar{Q}_{ij}$ matrix become:

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})\, m^2n^2 + Q_{22}n^4 \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\, m^2n^2 + Q_{12}(m^4 + n^4) \\
\bar{Q}_{13} &= (Q_{11} + Q_{12} + 2Q_{66})\, m^3n - (Q_{12} - Q_{22} + 2Q_{66})\, mn^3 \\
\bar{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})\, m^3n^2 + Q_{22}m^4 \\
\bar{Q}_{23} &= (-Q_{11} + Q_{12} + 2Q_{66})\, mn^3 - (Q_{12} - Q_{22} + 2Q_{66})\, m^3n \\
\bar{Q}_{33} &= (Q_{11} + Q_{22} - 2Q_{66})\, m^2n^2 + Q_{66}(m^2 + n^2)
\end{aligned} \tag{16}$$

A more convenient form of the transformed lamina stiffness, in terms of invariants as given by Tsai and Pagano[3] and used later in the computer analysis, is:

$$\begin{aligned}
\bar{Q}_{11} &= U_1 + U_2\cos(2\theta) + U_3\cos(4\theta) \\
\bar{Q}_{12} &= U_4 - U_3\cos(4\theta) \\
\bar{Q}_{13} &= \tfrac{1}{2}U_2\sin(2\theta) + U_3\sin(4\theta) \\
\bar{Q}_{22} &= U_1 - U_2\cos(2\theta) + U_3\cos(4\theta) \\
\bar{Q}_{23} &= \tfrac{1}{2}U_2\sin(2\theta) - U_3\sin(4\theta) \\
\bar{Q}_{33} &= U_5 - U_3\cos(4\theta)
\end{aligned} \tag{17}$$

where

$$\begin{aligned} U_1 &= \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}) \\ U_2 &= \frac{1}{2}(Q_{11} - Q_{22}) \\ U_3 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}) \\ U_4 &= \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}) \\ U_5 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66}) \end{aligned} \tag{18}$$

Laminated Plate Theory

Strain Displacement Relationships

A laminate is composed of several orthotropic layers. As such, the description of the behavior of a single lamina, as previously discussed, forms the basis or building block with which the behavior of a laminate may be described. Equation (14) gives the constitutive relationship for a lamina with respect to an arbitrary xy coordinate system. Considering the arbitrary xy axes to be oriented with the laminate axes, Equation (14) can be thought of as a stress-strain relationship for the k^{th} layer of a multi-layered laminate and may be written as

$$\{\sigma\}_k = [\bar{Q}]_k \{\varepsilon\}_k \tag{19}$$

Knowing the variation of stress and strain through the laminate thickness is essential to the definition of the extensional and bending stiffness of a laminate. The laminate is assumed to consist of layers of perfectly bonded laminae, such that the displacements are continuous across lamina boundaries and one lamina cannot slip

relative to another. With this assumption, and if the laminate is thin, it may be assumed that a line originally straight and perpendicular to the middle surface of the laminate will remain straight and perpendicular to the middle when the laminate is extended and bent.

Considering a section of laminate in the xy plane deformed due to some loading, as shown in Figure 3, the geometrical midplane undergoes some displacement, u_o, in the x-direction. With the above assumption, the line ABD remains straight and normal to the deformed

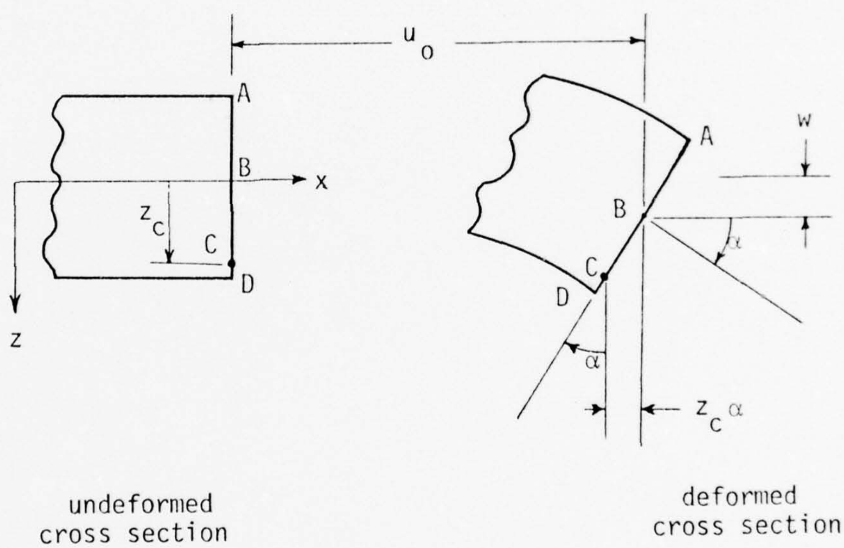


Figure 3. Geometry of deformation in the xz plane

midplane and the displacement in the x-direction of any point, C, on the normal ABD is given by the linear relationship[3]

$$u_c = u_o - z_c\alpha \tag{20}$$

where z_c is the z coordinate of the point C and α is the slope of ABD with respect to the original vertical line. Also, under deformation, line ABD remains perpendicular to the middle surface so that the slope of the laminate surface in the x-direction is

$$\alpha = \frac{\partial w}{\partial x} \tag{21}$$

where w is the displacement in the z-direction. Substituting Equation (21) into Equation (20), the displacement, u, at any point, z, through the laminate thickness is

$$u = u_o - z \frac{\partial w}{\partial x} \tag{22}$$

By similar reasoning, the displacement, v, in the y-direction is

$$v = v_o - z \frac{\partial w}{\partial y} \tag{23}$$

The assumptions thus far are equivalent to ignoring the shearing strains in planes perpendicular to the middle surface, that is, γ_{xz}, $\gamma_{yz} = 0$. Also, the line ABD is assumed to have constant length so that $\varepsilon_z = 0$.[1] These assumptions, known as the Kirchhoff-Love hypothesis, reduce the strain-displacement relationships to[1]

i.e. delamination ignored.

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \tag{24}$$

Differentiating Equations (22) and (23) and substituting into Equation (24), the strains become

$$\varepsilon_x = \frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2} \tag{25}$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - 2z\,\frac{\partial^2 w}{\partial x \partial y}$$

or

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{26}$$

where the middle surface strains are

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_o}{\partial x} \\ \dfrac{\partial v_o}{\partial y} \\ \dfrac{\partial v_o}{\partial y} + \dfrac{\partial v_o}{\partial x} \end{Bmatrix} \tag{27}$$

and the middle surface curvatures are[1]

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\,\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{28}$$

By substitution of the strain variation through the thickness, Equation (26) into Equation (19), the stresses in the k^{th} layer can be expressed in terms of the laminate middle surface strains and

curvatures as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_k \left\{ \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right\} \tag{29}$$

Laminate Constitutive Equations

Since the $\bar{Q}_{ij}$ can be different for each layer of the laminate, the stress variation through the laminate is not necessarily linear even though the strain variation is linear. To investigate these non-linear stresses, the resultant laminate forces and moments, denoted by N and M respectively, are obtained by integration of the stresses in each layer of lamina through the laminate thickness.[1] For example,

$$N_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \, dz$$

$$M_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \, z dz \tag{30}$$

where t is the total laminate thickness.

The total force and moment resultants for a n-layered laminate may then be defined as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}_k = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \tag{31}$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} zdz \tag{32}$$

or, using Equation (29) and summing over the laminate thickness, for a laminate with n layers,[1]

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_k \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} dz + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} zdz \right\} \tag{33}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_k \left\{ \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} zdz + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_z \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} z^2 dz \right\} \tag{34}$$

$[\varepsilon^o]$ and $[\kappa]$ are not functions of z, but are midplane values, so they can be removed from under the summation signs and Equations (33) and (34) can be written as[1]

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{35}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{23} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{36}$$

where

$$E_{ij} = \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \, (z_k - z_{k-1}) \tag{37}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \, (z_k^2 - z_{k-1}^2) \tag{38}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \, (z_k^3 - z_{k-1}^3) \tag{39}$$

For laminates that are symmetric in both geometry and material properties about the middle surface, Equations (35) and (36) simplify considerably. In particular, because of the symmetry of $[\bar{Q}_{ij}]_k$ and the lamina thickness t_k, all the $[B_{ij}]$ are equal to zero and the force and moment resultants for a symmetric laminate are

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \tag{40}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{41}$$

For the remainder of this investigation, the laminate will be considered to be symmetrical and in a state of pure tension or compression, that is, bending moments will be zero and only Equation (40) will apply.

Ply Failure Criteria

Modified Maximum Strain Theory

Most experimental determinations of the strength of a material are based on uniaxial stress states. However, the practical problem usually involves at least a biaxial state of stress. For an orthotropic lamina, strength criteria parallel and transverse to the fiber direction due to tension, compression and shear strength may all be experimentally determined. To relate this uniaxial strength information to an analysis of ply damage and progressive failure, the following modified maximum strain theory is proposed. The lamina is said to have failed in the fiber direction if

$$\varepsilon_1 > X_{\varepsilon t} \quad \text{or} \quad \varepsilon_1 < X_{\varepsilon c} \tag{42}$$

and transverse to the fibers if

$$\varepsilon_2 > Y_{\varepsilon t} \quad \text{or} \quad \varepsilon_2 < Y_{\varepsilon c} \tag{43}$$

where $X_{\varepsilon t}$, $X_{\varepsilon c}$, and $Y_{\varepsilon t}$, $Y_{\varepsilon c}$ indicate the maximum allowable tensile and compressive strains in the 1 and 2 directions. In the same way, the lamina is said to have failed in shear if

$$|\gamma_{12}| > S_\varepsilon \tag{44}$$

where S_ε is the maximum allowable shear strain.

This failure theory makes it possible to obtain post-failure constitutive equations. However, stress or strain interactions, such as the combined effect of transverse strain and shear on failure, have been ignored.

Post Failure Constitutive Equations

In forming the modified strain theory, the following assumptions were made:

(1) If a lamina fails in the fiber direction, the matrix will still carry a load transverse to the fibers, but will not carry a shear load.

(2) If a lamina fails transverse to the fiber direction, it will not carry a shear load, but the fibers will carry a normal load parallel to the fibers.

(3) If a lamina fails in shear, the matrix will not carry a load transverse to the fibers, but the fibers will carry a normal load.

(4) If a lamina fails in the fiber direction and in shear or fails both parallel and transverse to the fibers, the lamina is considered to have totally failed and will not support a load.

Each of the above assumptions indicates a partial or total failure of the lamina. Examination of Equation (4) shows that for a partial failure of a lamina at a given strain, the stress is changed by a change or softening of the stiffness matrix. In the computer solution of the biaxial stress problem, the changes in stiffness and load are used. Therefore, the post-failure lamina constitutive equations will be expressed in terms of change in stress and stiffness due to partial or total lamina failure.

Using Equation (4), the change in stress due to a change in stiffness is given by

$$\begin{Bmatrix} \Delta\sigma_1 \\ \Delta\sigma_2 \\ \Delta\tau_{12} \end{Bmatrix} = [\Delta Q_{ij}] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{45}$$

where Q_{ij} is the change in stiffness due to failure, or

$$\Delta Q_{ij} = Q_{ij} - \text{post failure stiffness} \tag{46}$$

The ΔQ_{ij} terms for the various types of lamina failure are found as follows:

(1) Lamina failure in the fiber direction is assumed to cause E_{11}, G_{12}, ν_{12}, to equal zero leaving E_{22} as the only factor contributing to the new Q_{ij} stiffness. Using Equations (5) and (46), the ΔQ_{ij} terms are found to be

$$\begin{aligned} \Delta Q_{11} &= \frac{E_{11}}{(1 - \nu_{12}\nu_{21})} \\ \Delta Q_{22} &= \frac{E_{22}(\nu_{12}\nu_{21})}{(1 - \nu_{12}\nu_{21})} \\ \Delta Q_{12} &= \frac{\nu_{12} E_{11}}{(1 - \nu_{12}\nu_{21})} \\ \Delta Q_{66} &= G_{12} \end{aligned} \tag{47}$$

(2) In the same way, lamina failure transverse to the fibers is assumed to cause E_{22}, G_{12}, ν_{12} to equal zero leaving E_{11} as the only contributing stiffness factor. For this type failure the ΔQ_{ij} terms are

$$\Delta Q_{11} = \frac{E_{11}(\nu_{12}\nu_{21})}{(1 - \nu_{12}\nu_{21})}$$

$$\Delta Q_{11} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} \qquad (48)$$

$$\Delta Q_{12} = \frac{\nu_{12} E_{11}}{(1 - \nu_{12}\nu_{21})}$$

$$\Delta Q_{66} = G_{12}$$

(3) Lamina failure in shear is assumed to cause E_{22}, G_{12}, ν_{12}, to equal zero leaving E_{11} as the only contributing stiffness factor. Thus, Equation (49) also gives the ΔQ_{ij} terms for shear failure.

(4) Lamina failure in the fiber direction and in shear, or failure both parallel and transverse to the fibers, is assumed to cause total lamina failure and therefore, zero remaining stiffness. The ΔQ_{ij} terms obtained by Equations (5) and (46) are

$$\Delta Q_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}$$

$$\Delta Q_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} \qquad (49)$$

$$\Delta Q_{12} = \frac{\nu_{12} E_{11}}{(1 - \nu_{12}\nu_{21})}$$

$$\Delta Q_{66} = G_{12}$$

Denoting ΔQ_{ij} as the change in lamina stiffness with respect to the arbitrary xy axes and using Equations (8), (10), and (45), the change in stress in the xy coordinate system due to a change in stiffness is given by

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\tau_{xy} \end{Bmatrix}_k = [\Delta\bar{Q}_{ij}]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \tag{50}$$

where

$$[\Delta\bar{Q}_{ij}]_k = [T_{ij}]_k \, [\Delta Q_{ij}]_k \, [T_{ij}]_k^{-1} \tag{51}$$

The ΔQ_{ij} terms are calculated by use of Equations (17) and (18) where ΔQ_{ij} terms are substituted for Q_{ij} terms.

Laminate Failure Criteria

With laminate strength, just as with the determination of laminate stiffness, the basic building block is the lamina with its inherent characteristics. Basic to determining the strength of a laminate is a knowledge of the stress state in each lamina. However, failure of one layer does not necessarily imply failure of the entire laminate. The laminate may, in fact, be capable of higher loads despite a significant change in stiffness.

The strength of an angle-ply laminate, symmetric about its middle surface, may be determined by examining the state of damage in each layer for a particular load. The laminate strains are calculated from the known load and stiffness prior to failure of a lamina. If one or more lamina have failed, as determined from the failure criterion, a new laminate stiffness is calculated and the laminate strains recalculated to determine the post-failure strains. Then it must be verified that the remaining laminae, at their increased strain levels, do not

fail at this applied load. Should an applied load cause progressive failure, where all layers successfully fail at the same load, the laminate is said to have suffered gross failure.[1]

An alternative method, described in the next section, uses the original laminate stiffness to determine the strains at each load or failure cycle. When a failure takes place, a change in stiffness due to the failure is calculated. Using the change in stiffness and the known strains, a pseudo load is calculated and added to the original load, giving the required increase in strain. In an iterative finite element program this method is useful in that the stiffness matrix is only inverted once.

Laminate Post-Failure Constitutive Equations

The strength of a symmetric angle-ply laminate subjected to plane stress is determined by first finding the strains for a known load. Inverting the stiffness matrix, Equation (40) can be written

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = [E_{ij}]^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \tag{52}$$

From the previous assumption that plane sections perpendicular to the midplane axis remain plane, and for a state of plane stress, Equation (52) gives the state of strain for all layers. Then, by Equation (8), the strain with respect to the 1-2 axes for each layer k, is

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix}_k = [T_{ij}]_k \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \frac{\gamma_{xy}^o}{2} \end{Bmatrix} \tag{53}$$

The lamina strains are compared with the lamina failure criteria, Equations (42), (43), and (44), to determine modes of failure.

Should failures occur, changes in stiffness for each layer are calculated using Equations (47), (48), or (49), depending on the type of failure, and Equation (51). The total change in laminate stiffness is found by summing the laminar stiffness changes. For an n ply laminate, the total change in stiffness, ΔE_{ij}, is

$$[\Delta E_{ij}] = \sum_{k=1}^{n} [\Delta \bar{Q}_{ij}]_k \, (z_k - z_{k-1}) \tag{54}$$

where $\Delta\bar{Q}_{ij}$ is from Equation (51) and $z_k - z_{k-1}$ is the thickness of lamina n.

Knowing the change in stiffness and the laminate strains, a pseudo force, PN, due to the loss of stiffness is found by

$$\begin{Bmatrix} PN_x \\ PN_y \\ PN_{xy} \end{Bmatrix} = [\Delta E_{ij}] \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} \tag{55}$$

Adding this pseudo force to the applied load of Equation (52) gives the increased strain due to lamina failure. Then, the new strain is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [E]^{-1} \begin{Bmatrix} N_x + PN_x \\ N_y + PN_x \\ N_{xy} + PN_{xy} \end{Bmatrix} \tag{56}$$

Equations (53) through (56) are repeated until equilibrium is obtained or the laminate experiences gross failure.

Application to Composite Structures

Thus far in this thesis, lamina and laminate have been analyzed in a state of plane stress, but the geometry and boundary conditions have not been considered. It has been assumed that the state of strain is constant throughout the laminate and that the stress is constant throughout each layer. In actual applications the stress in a laminate and in a lamina may vary considerably due to geometry and loading conditions. Stress concentrations such as holes, notches and cracks may increase the local stress to a much greater value than the stress at another point in the member. Such localized stresses may lead to localized laminate failure and ultimately to complete laminate failure at reduced loads.

In order to analyze a varying state of stress at points across a laminate, the finite element method will be used in conjunction with the previous ply and laminate equations.

NUMERICAL PROCEDURE

Finite Element Method For Plane Stress Analysis

In a matrix analysis of composite materials the standard approach is to divide the composite laminate into a finite number of elements connected at joints or nodal points. The stiffness or flexibility properties of each individual element are then established by an element analysis, and the element stiffnesses combined to form the stiffness matrix for the complete structure. In the discussion that follows, a brief description of the displacement method for a constant strain triangle element will be presented and then incorporated with the previous ply and laminate equations in an iteration method to provide a solution to the nonlinear composite laminate problem.

Figure 4 depicts a typical triangular element with nodes i, j, and m, numbered in counter-clockwise order. Each node may have displacements in the x and y directions. Then, denoting displacements in the x and y directions by u and v respectively, the six components of element displacement may be written as the vector $\{\delta\}^e$ where

$$\{\delta\}^e = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \tag{57}$$

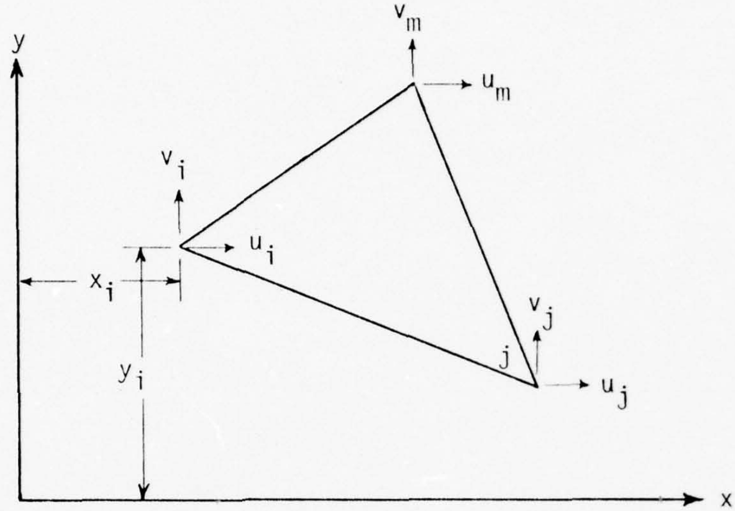


Figure 4. Plane Stress Triangular Element

The displacement within an element have to be uniquely defined by these six displacement values. Representing the displacements by two linear polynomials[23]

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \tag{58}$$

the nodal displacements can be written

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$
$$u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \tag{59}$$
$$u_m = \alpha_1 + \alpha_2 x_m + \alpha_3 y_m$$

$$v_i = \alpha_4 + \alpha_5 x_i + \alpha_6 y_i$$

$$v_j = \alpha_4 + \alpha_5 x_j + \alpha_6 y_j \tag{60}$$

$$v_m = \alpha_4 + \alpha_5 x_m + \alpha_6 y_m$$

Evaluating the six constants α in terms of the nodal displacements, gives[23]

$$u = \frac{1}{2\Delta}(a_i+b_ix+c_iy)u_i + (a_j+b_jx+c_jy)u_j + (a_m+b_mx+c_my)u_m \tag{61}$$

and

$$v = \frac{1}{2\Delta}(a_i+b_ix+c_iy)v_i + (a_i+b_jx+c_jy)v_j + (a_m+b_mx+c_my)v_m \tag{62}$$

where

$$a_i = x_jy_m - x_my_i$$

$$b_i = y_j - y_m \tag{63}$$

$$c_i = x_m - x_j$$

and

$$2\Delta = \det \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = 2 \text{ (area of triangle ijm)} \tag{64}$$

Neglecting any initial strain, the total strain at any point within the element can be defined by its three components that contribute to the internal work. From Equations (24)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{65}$$

Using Equations (57), (61), (62), and (65), the strain within the triangular element expressed in terms of nodal displacement is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^e = [B]^e \{\delta\}^e \tag{66}$$

where

$$[B] = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \tag{67}$$

and the B terms are

$$\begin{array}{ll} b_i = y_i - y_m & c_i = x_m - x_j \\ b_j = y_m = y_i & c_j = x_i - x_m \\ b_m = y_i - y_j & c_m = x_j - x_i \end{array} \tag{68}$$

By Equation (40), the stress resultant within an anisotropic element is

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}^e = [E] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^e = [E][B]^e \{\delta\}^e \tag{69}$$

Nodal forces can also be expressed in terms of the nodal displacements. Denoting U and V as the nodal forces in the x and y directions and assuming zero body forces, the nodal force vector $\{F\}^e$ for a triangular element can be written

$$\{F\}^e = \begin{Bmatrix} U_i \\ V_i \\ U_j \\ V_j \\ U_m \\ V_m \end{Bmatrix} \tag{70}$$

The stresses that result from these nodal forces can be found by equating the work done by the forces to the strain energy stored in the element. The work done by the nodal forces is[23]

$$W = \frac{1}{2}\,(U_i u_i + V_i v_i + U_j u_j + V_j v_j + U_m u_m + V_m v_m) \tag{71}$$

or

$$W = \frac{1}{2}\,\{F\}^{e^T}\,\{\delta\}^e \tag{72}$$

The strain energy is given by[23]

$$\begin{aligned} \bar{U} &= \frac{t}{2}\iint(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})dA \\ &= \frac{t}{2}\{\sigma\}^{e^T}\,\{\varepsilon\}^e\,\Delta \end{aligned} \tag{73}$$

where Δ is the area of the triangular element and t is the element thickness. Using Equations (66) and (69), the strain energy can be

rewritten

$$\bar{U} = \frac{t\Delta}{2}\{\delta\}^{e^T}[B]^T[E]^T[B]\{\delta\}^e \tag{74}$$

Equating the work and energy equations and taking the transpose of both sides,

$$\{F\}^e = \Delta t\ [B]^T[E][B]\{\delta\}^e \tag{75}$$

Denoting the element stiffness matrix $[K]^e$,

$$\{F\}^e = [K]^e\{\delta\}^e \tag{76}$$

and

$$[K]^e = \Delta t\ [B]^T[E][B] \tag{77}$$

Equations (76) and (77) are now sufficient for computation with the actual matrix operations being accomplished in the computer program. Combining the element stiffness matrices and their force and displacement vectors gives the structural system of equations

$$[A]\ \{\delta\} = \{F\} \tag{78}$$

where [A] is the structural stiffness matrix.

Solution Method for Nonlinear Material Properties

Initial Stress Process

The expressions derived in the previous sections describe fully the stress-strain relations for a laminated composite material in a state of plane stress. The essential nonlinearity is evident from

Equations (54), (55), and (56) with the composite stiffness matrix being dependent on the state of total stress. This problem, as described in the following section, can be approached using peacewise linearization to obtain a solution iteratively.[18,26]

The "initial stress" process approaches the solution of a nonlinear problem as a series of approximations.[15-19,20-25] In the first step after a load increment a purely elastic problem is solved determining an increment of strain $\{\Delta\varepsilon'\}$ and of stress $\{\Delta\sigma'\}$ at every point of the structure. The nonlinearity implies that for the increment of strain found, the increment of stress will, in general, not be correct. If the true increment of stress for equilibrium is $\{\Delta\sigma\}$, then the correct solution can be maintained by a set of pseudo body forces equilibrating the "initial stress" system $\{\Delta\sigma'\} - \{\Delta\sigma\}$.[19]

At the second stage of computation the system of pseudo body forces can be removed by allowing the structure (with unchanged elastic properties) to deform further. An additional set of strain and stress increments is caused, and once again they are likely to exceed those permitted by the nonlinear problem. The redistribution of pseudo body forces is repeated and the process continued until it converges to the nonlinear equilibrium conditions.

Application to Composite Materials

In laminated composite materials, the nonlinearity comes from failure or partial failure of a ply within a laminate. Ply failure or partial failure implies that a change in stiffness has taken place and that the load used and displacements found, for an elastic solution, are not correct. To arrive at the correct solution, pseudo body

forces are calculated using the change in stiffness and the laminate strains. These pseudo forces are allowed to further deform the laminate using the original elastic properties. New strains are found and the process repeated until equilibrium is obtained.

Specific steps in the initial stress process as applied to composite materials are:

(1) The problem is set up by using Equation (77), for each element, to construct the structural stiffness matrix. An incremental load and other boundary conditions are entered into Equation (78) which gives the following system of equations to be solved:

$$[A]\{\delta\} = \{F\} \tag{79}$$

(2) The stiffness matrix [A] is partially inverted and the displacement $\{\delta\}$ computed.

(3) Strains within each element are found by Equation (66)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^e = [B]^e\{\delta\}^e \tag{80}$$

(4) From Equation (8), the principal strains in each lamina are obtained,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix}^e_k = [T]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix}^e \tag{81}$$

(5) Lamina failure criteria, Equations (42), (43), and (44) are applied. If there are no failures, go to step 10.

(6) The change in element stiffness due to failure is computed using Equation (54),

$$[\Delta E] = \sum_{k=1}^{n} [\Delta\bar{Q}]_k \, (z_k - z_{k-1}) \tag{82}$$

(7) Pseudo forces at each node point are found using $[\Delta E]$ as the element stiffness and Equations (66) and (75). Denoting the pseudo forces PF,

$$\{PF\} = [B]^T[\Delta E] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \Delta t \tag{83}$$

(8) *The maximum pseudo force*, PF_{max}, is compared with an accuracy constant ACC. If PF_{max} is less than ACC, equilibrium has been reached, go to step 10.

(9) Using the pseudo forces and the partially inverted stiffness matrix of step 2, additional displacements are found and added to the original displacements. Return to step 3.

(10) The load is incremented by adding the displacements obtained for an incremental load to existing displacements and returning to step 3.

Should the iteration process of steps 3 through 9 be repeated 20 times within an increment without reaching equilibrium, the laminate is considered to have suffered gross failure and the process is stopped.

SOLUTION OF PROBLEMS

Uniaxial Tension Specimens

To check the reliability of the modified strain theory and the finite element program, the predicted stress-strain curves and failure loads for $(0°)_s$, $(90°)_s$, $(0°/90°/90°/0°)_s$, $(90°/\pm 45°/0°)_s$ and $(90°/\pm 45°/90°)_s$ laminates were obtained for uniaxial tension loads. Figure 5 shows the finite element grid used. The 12 element grid, scaled to 5 inches (12.7 cm) in length and 1 inch (2.54 cm) in width, was loaded using incremental displacements in the direction shown. Zero displacement conditions were specified for nodes opposite the load end in the load direction and along one side transverse to the load direction.

To establish a stress-strain relationship for comparison to experimental data, one element was chosen and its state of stress and strain written out at the end of each increment. The failure status of eacy ply within each element, nodal displacements, iterations and the maximum pseudo force for each iteration were also written out.

Material properties used were those for Thornel 300/5208 graphite-epoxy, listed in Appendix A.

Circular Hole Specimens

Two laminates containing circular holes and loaded in uniaxial tension were investigated using the finite element grid shown in

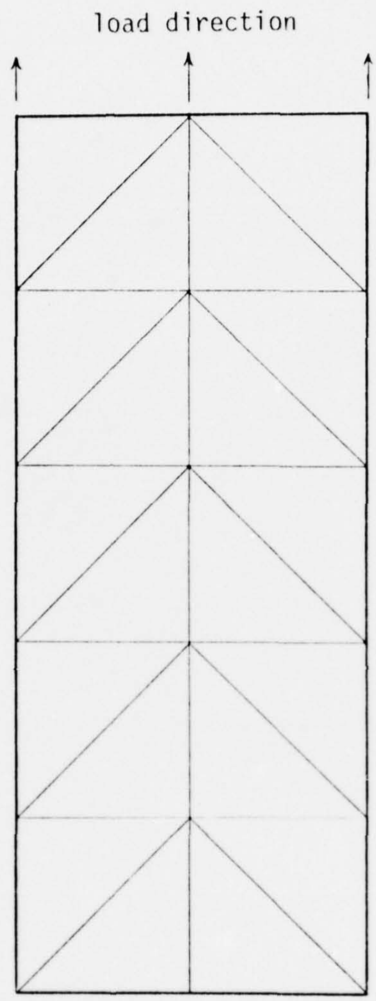


Figure 5. Finite element grid, uniaxial tension test

Figure 6. The 99 element grid was given an incremental displacement load in the direction shown, with zero displacement conditions imposed on the end opposite the load in the load direction and along the hole side transverse to the load direction. The scale for the grid represented a specimen 3 inches (7.62 cm) wide with a hole 1 inch (2.54 cm) in diameter.

At the end of each increment the status of eacy ply within each element was printed out. Total failure loads were determined using the nodal displacements and stiffnesses at failure to calculate the nodal forces at the load points.

Material properties used were those for Thornel 300/5208 graphite-epoxy, listed in Appendix A.

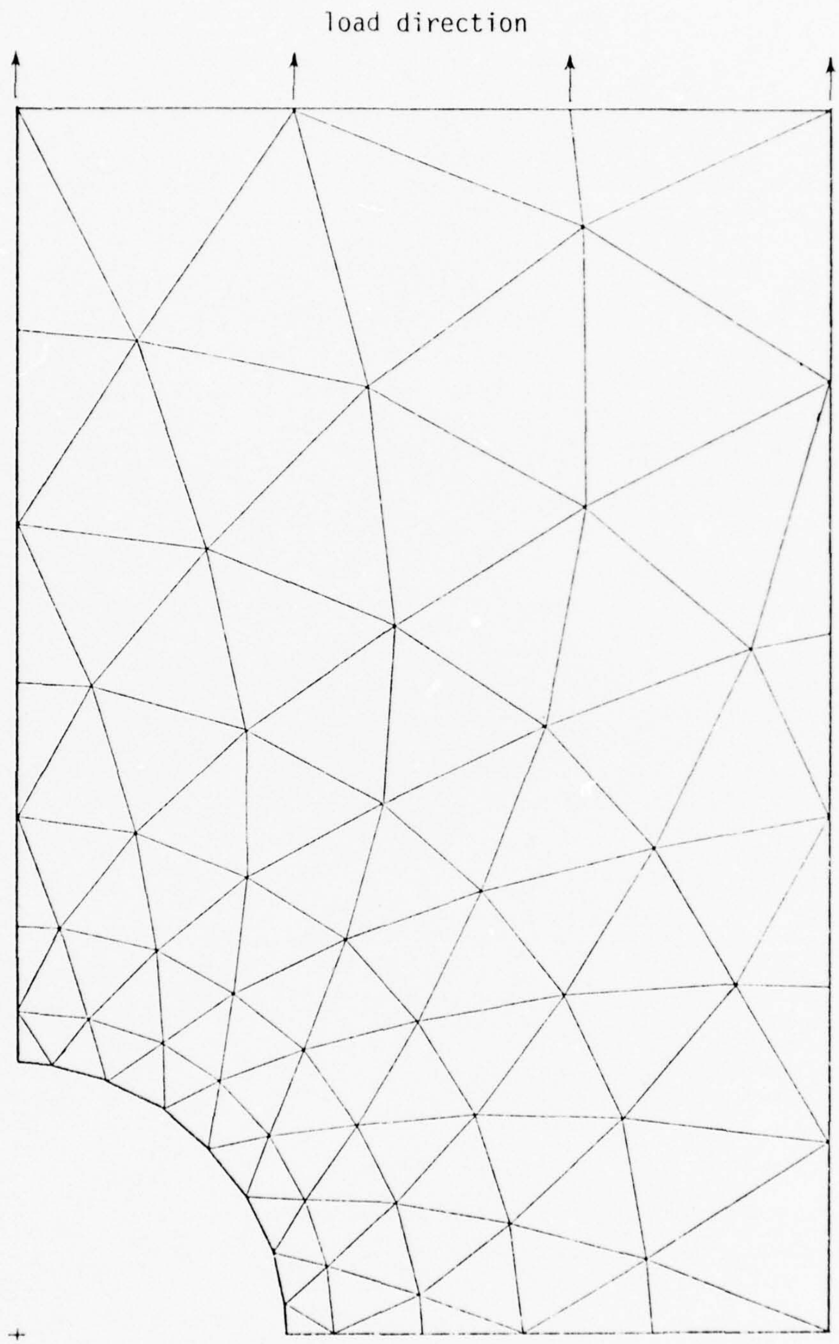


Figure 6. Finite element grid, circular hole in uniaxial tension

RESULTS

Uniaxial Tension Specimens

Stress vs. strain diagrams for the uniaxial tension problems are given in Figures 7 through 11. Data points plotted are those obtained by Sendeckyj for Thornell 300/5208 graphite-epoxy laminates.[29] The solid line is the predicted stress-strain curve as obtained from a chosen element. Changes in nodal displacements written out at the end of each increment were equal for all elements, indicating the strains, and thus the stresses, were equal for all elements.

Circular Hole Specimens

$(0°/90°/90°/0°)_S$ Laminate

Figures 12 through 16 give the damage or progressive failure status of the $(0°/90°/90°/0°)_S$ laminate at the end of each load increment. The first number code indicates the status of the 0° plies and the second number the status of the 90° plies. The specific numbers give the mode of failure within the ply.

The total failure load calculated for this notched laminate was 37,600 psi (2.6×10^8 Pa) while that obtained experimentally by Nuismer and Whitney was 28,200 psi (1.9×10^8 Pa).[30]

~30% higher

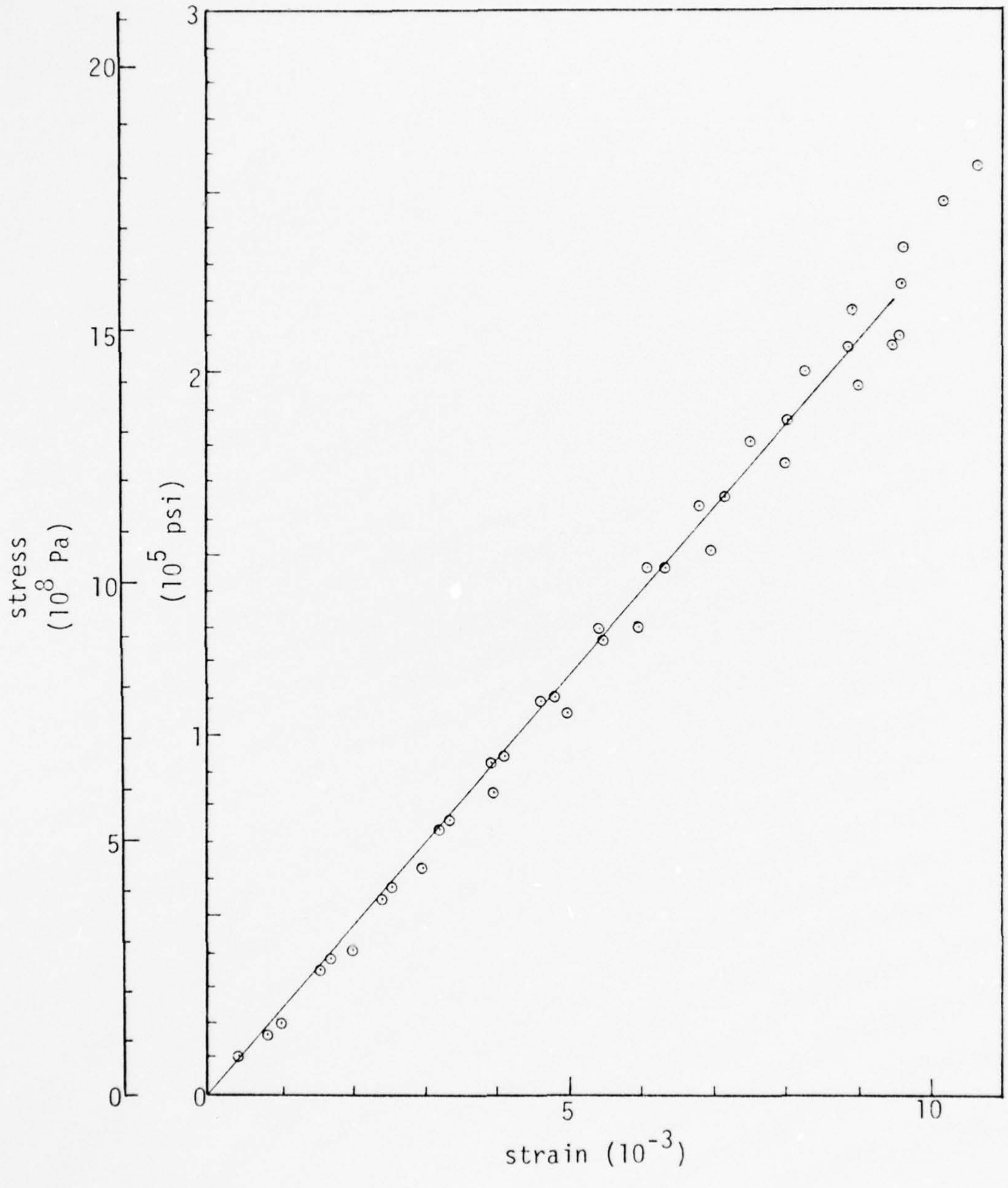


Figure 7. Stress vs strain, $(0^\circ)_s$ laminate in uniaxial tension

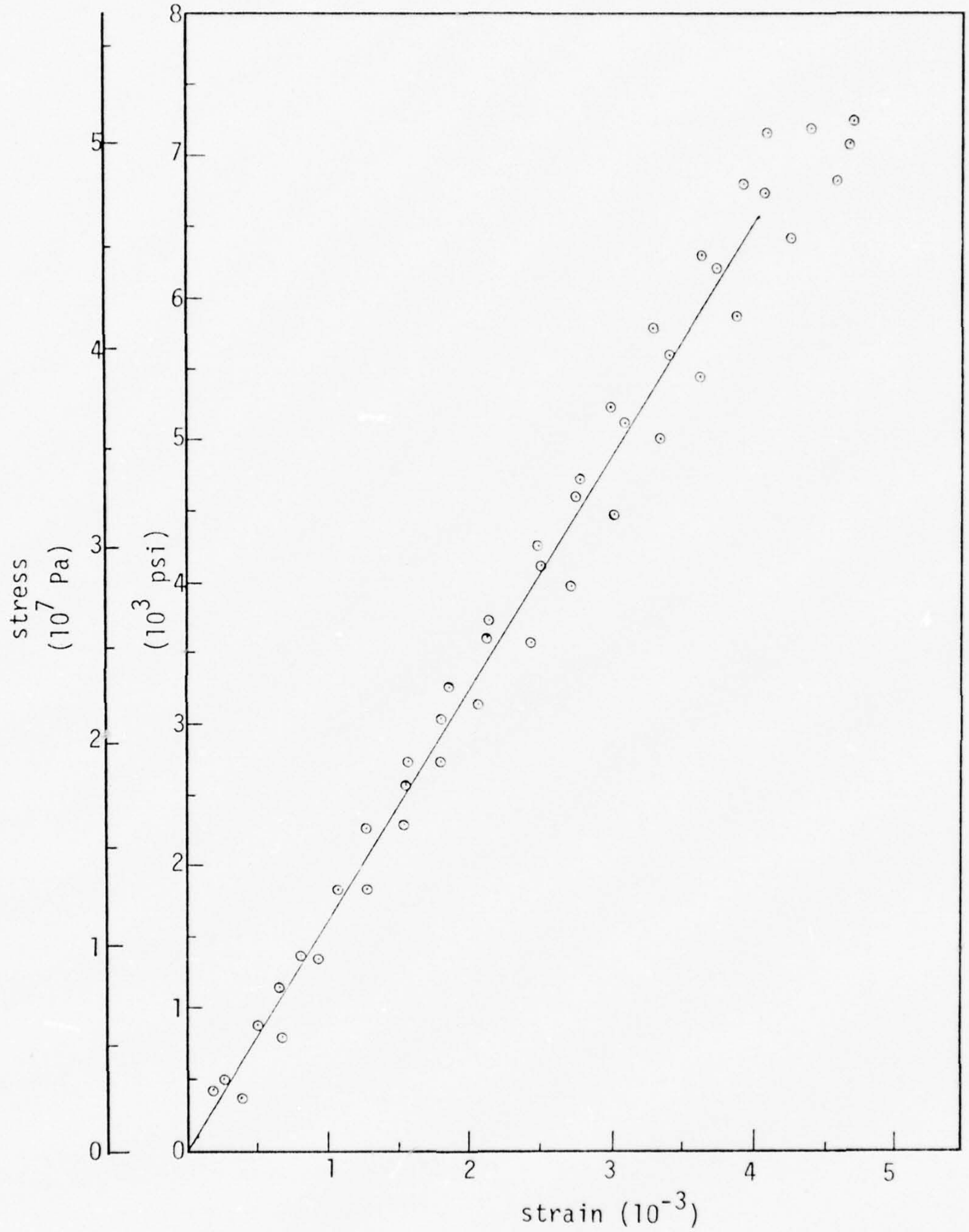


Figure 8. Stress vs strain, $(90°)_s$ laminate in uniaxial tension

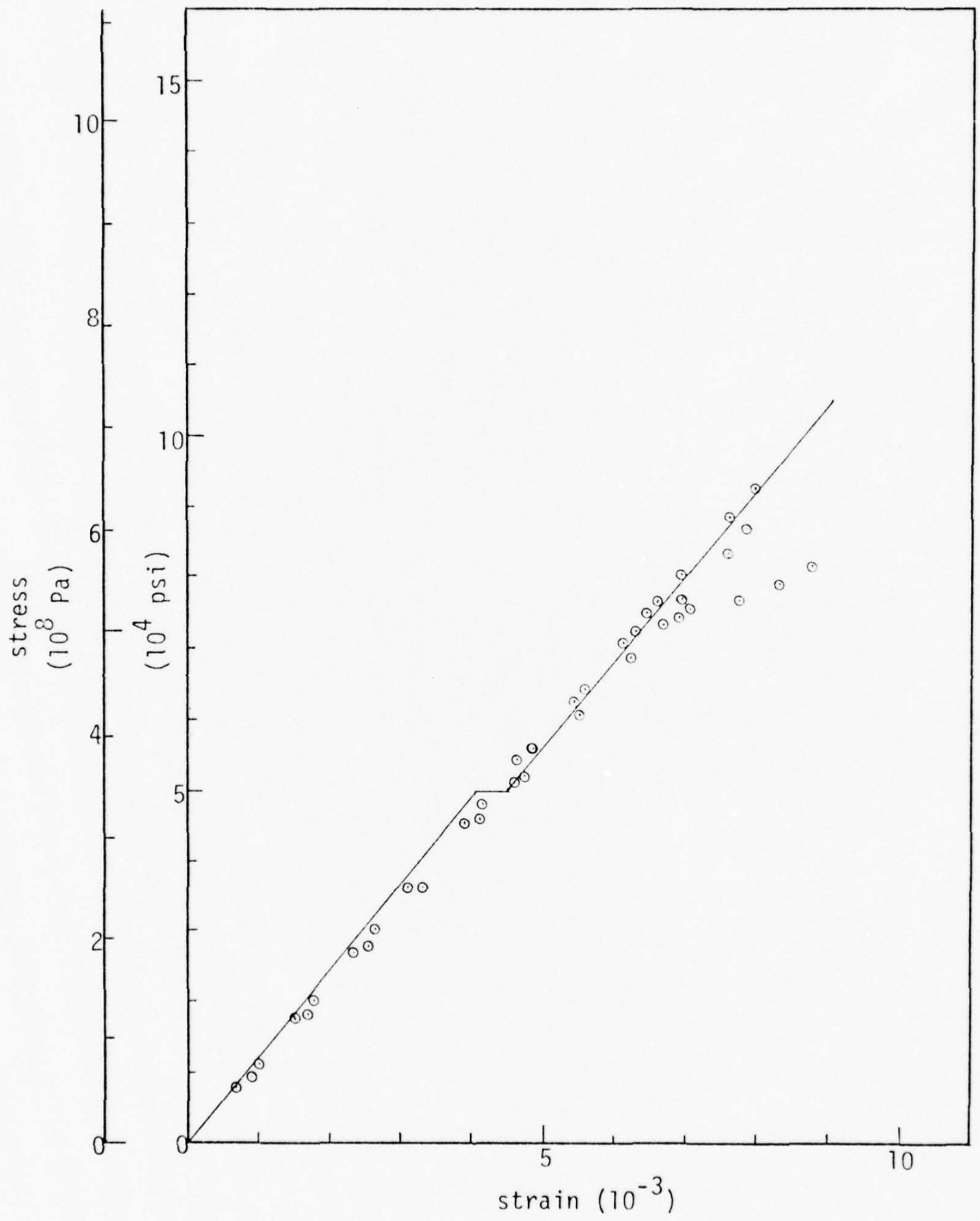


Figure 9. Stress vs strain, $(0°/90°/90°/0°)_s$ laminate in uniaxial tension

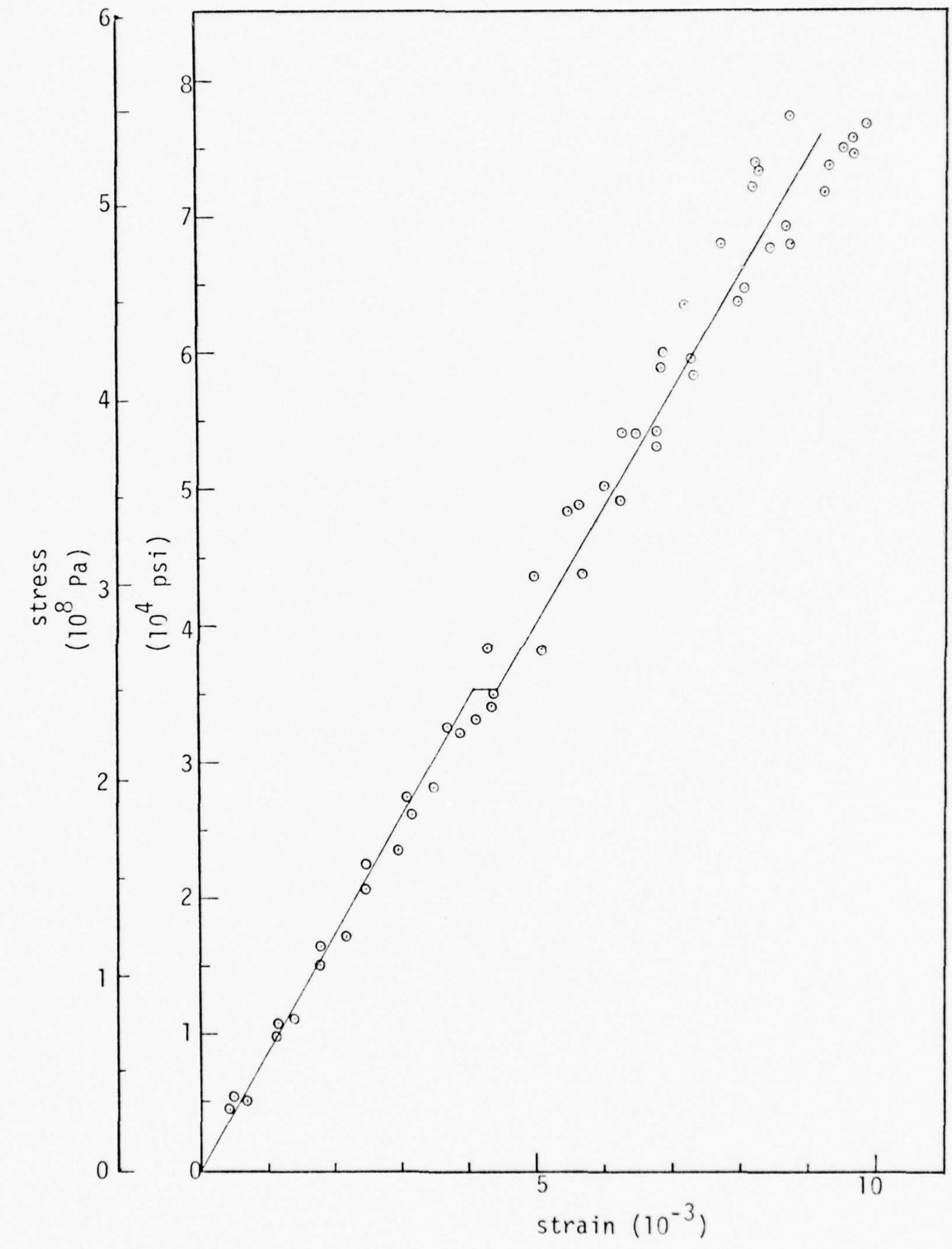


Figure 10. Stress vs strain, $(90°/\pm 45°/0°)_s$ laminate in uniaxial tension

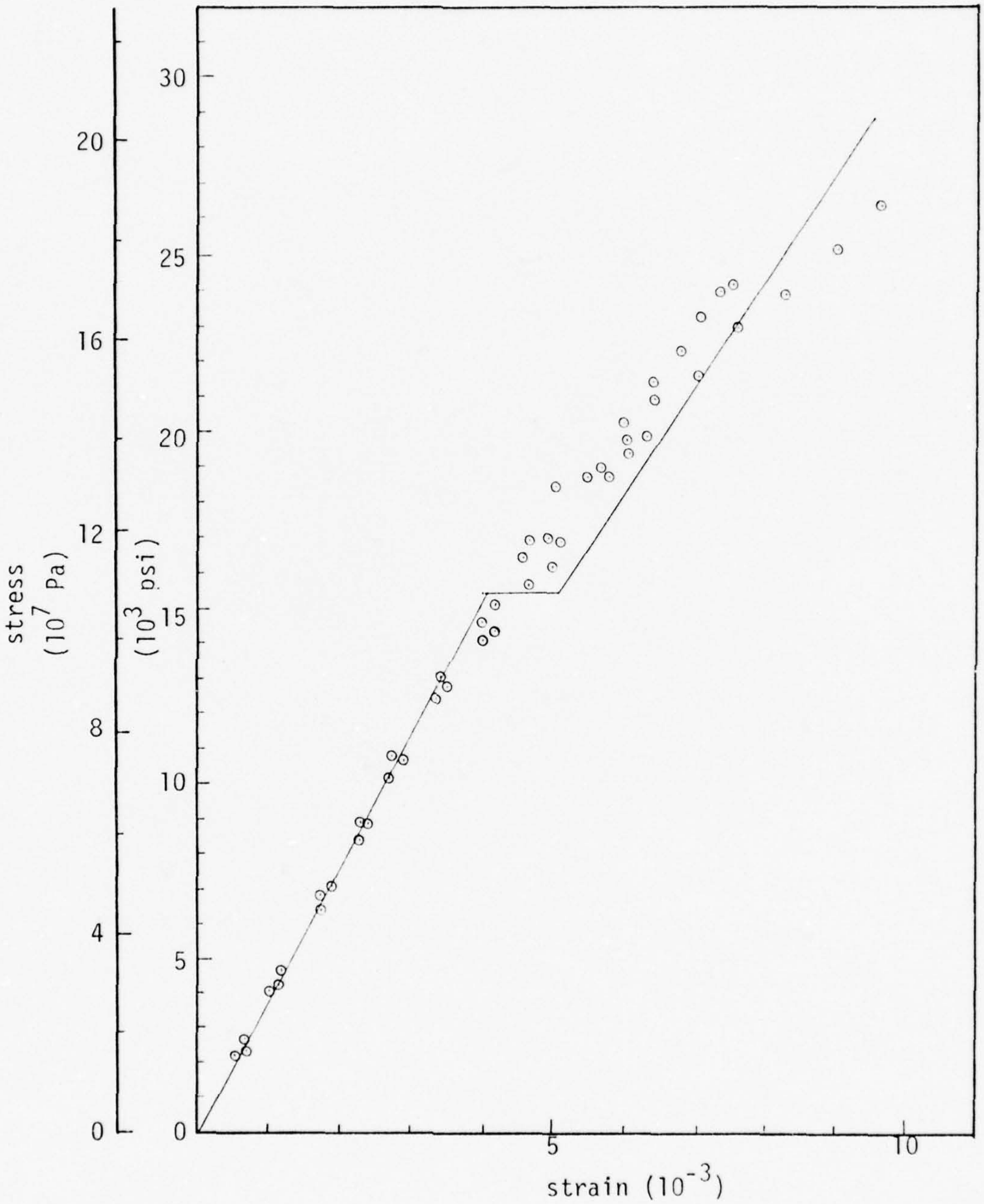
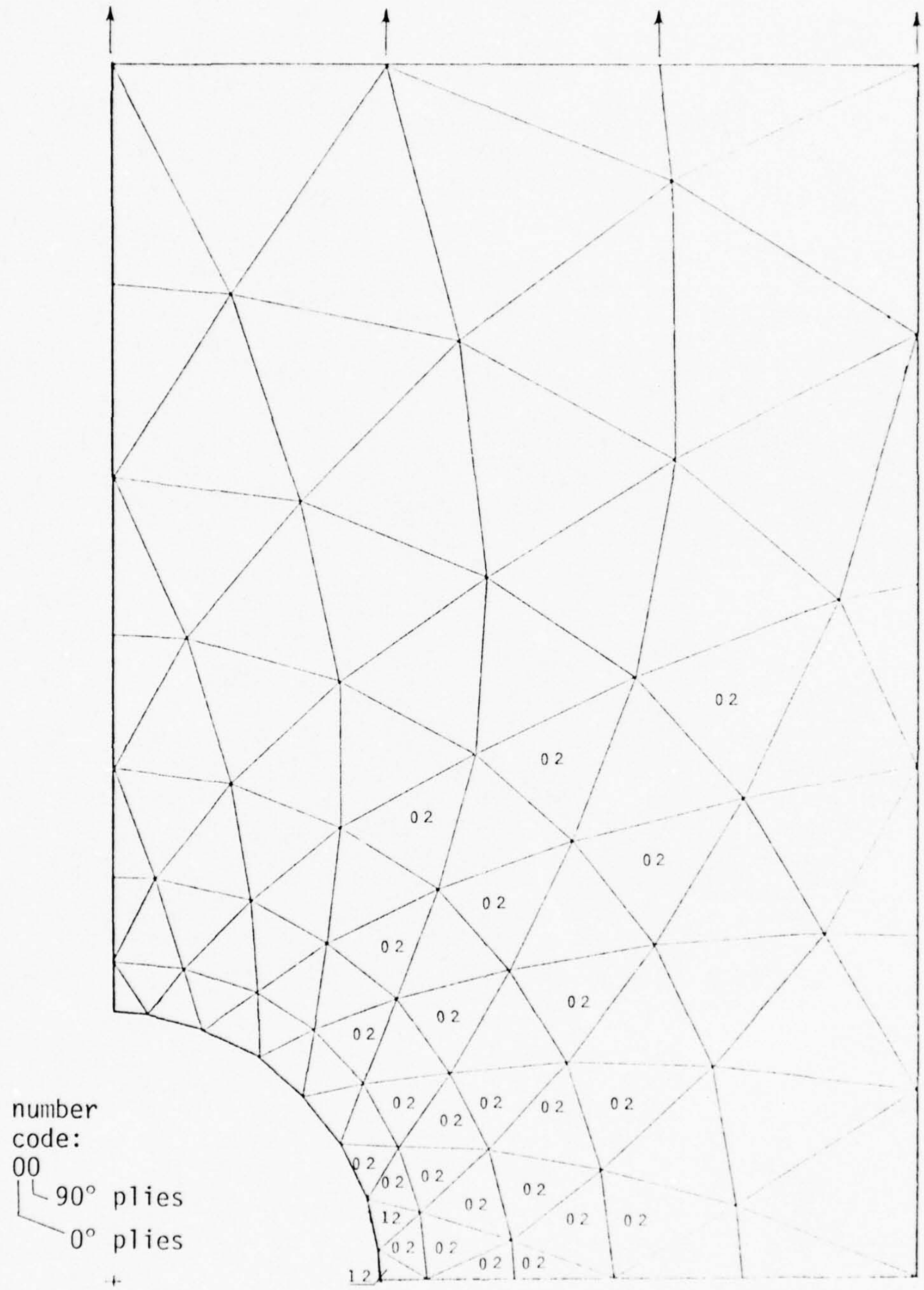


Figure 11. Stress vs strain $(90°/\pm 45°/90°)_s$ laminate in uniaxial tension



0 = no failure
1 = fiber failure
2 = matrix failure
3 = matrix and fiber failure

Figure 12. $(0°/90°/90°/0°)_s$ circular hole partial failure at .009 in. (.023 cm) displacement load

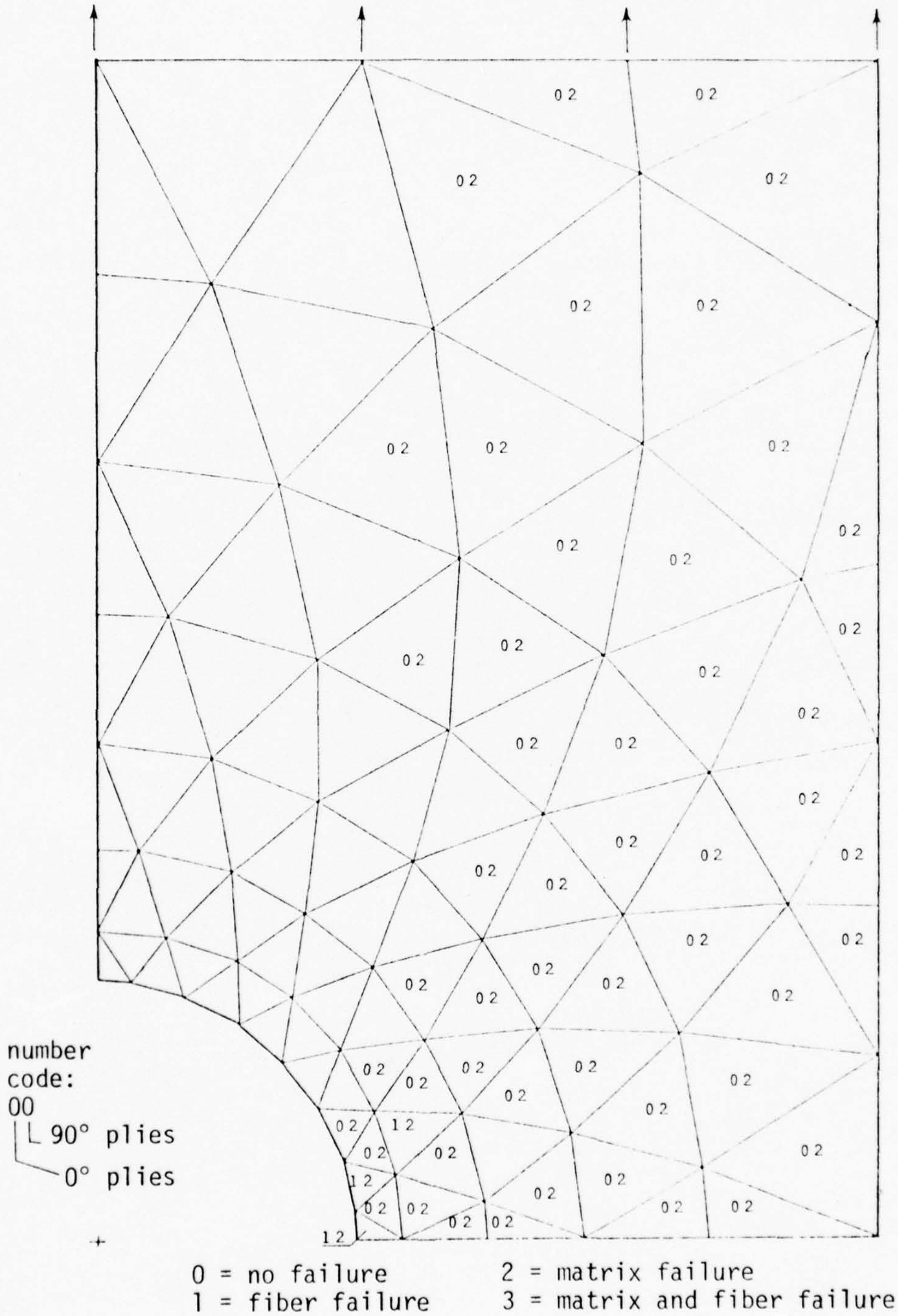
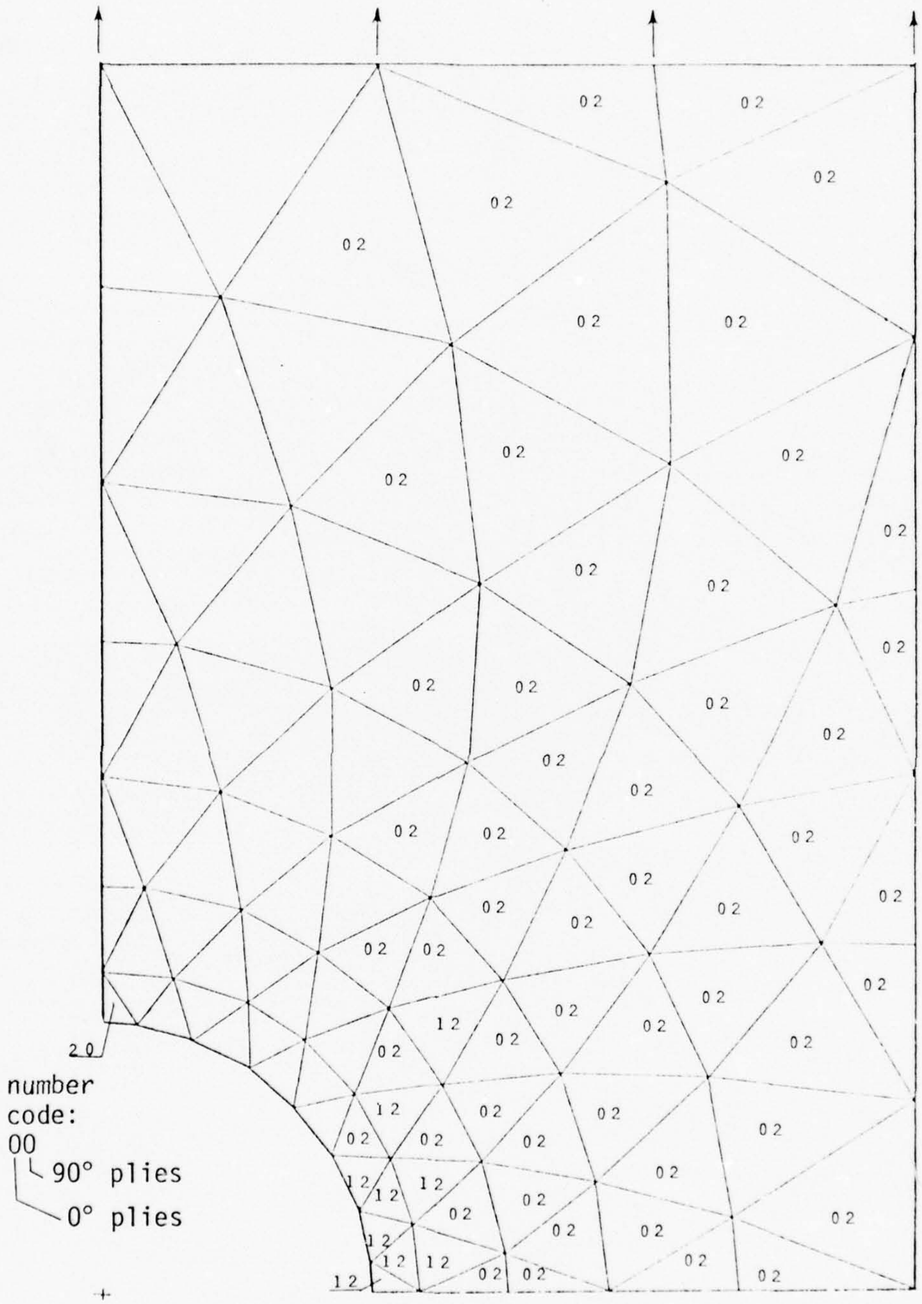
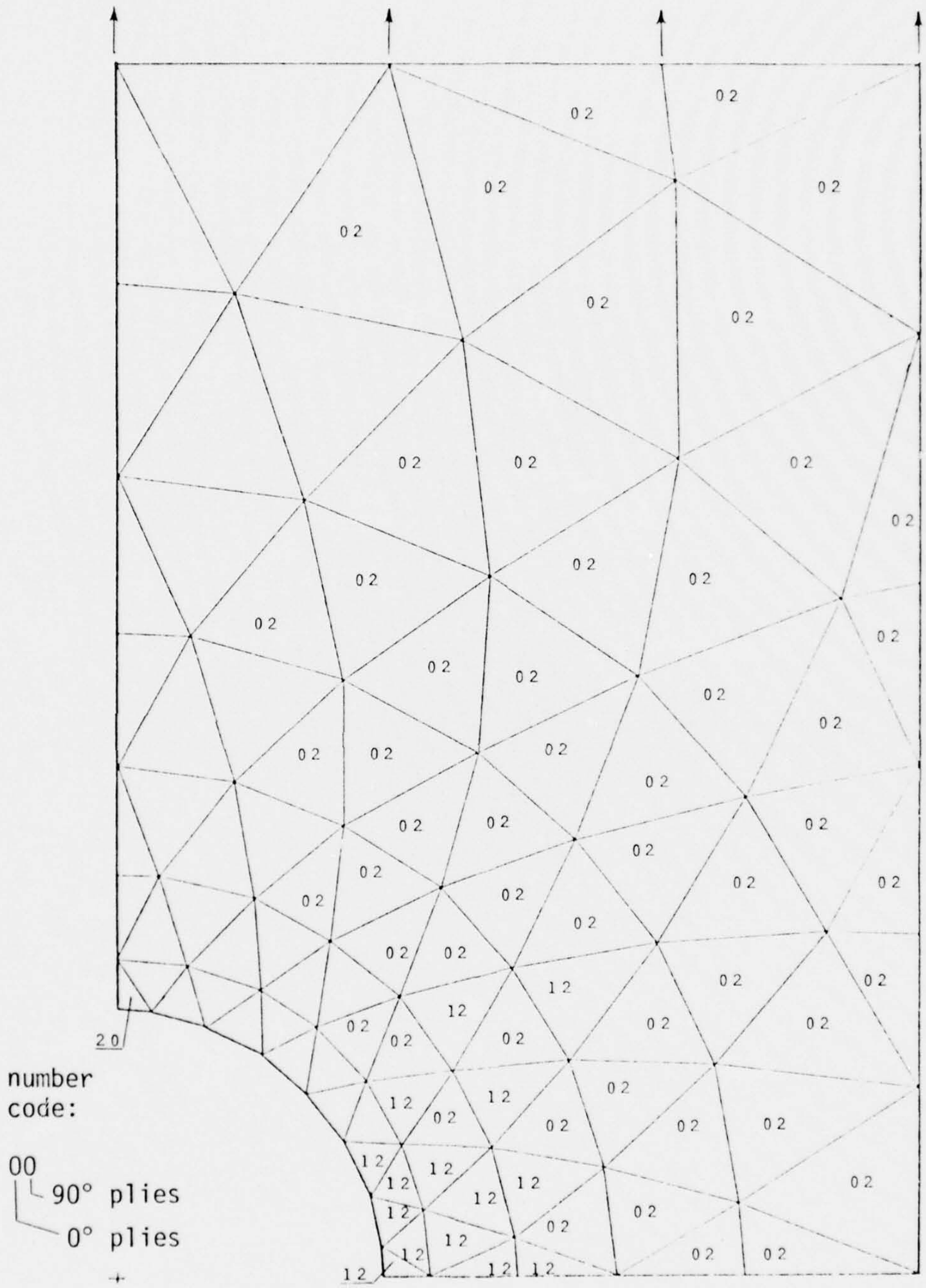


Figure 13. $(0°/90°/90°/0°)_s$ circular hole partial failure at .012 in. (.030 cm) displacement load



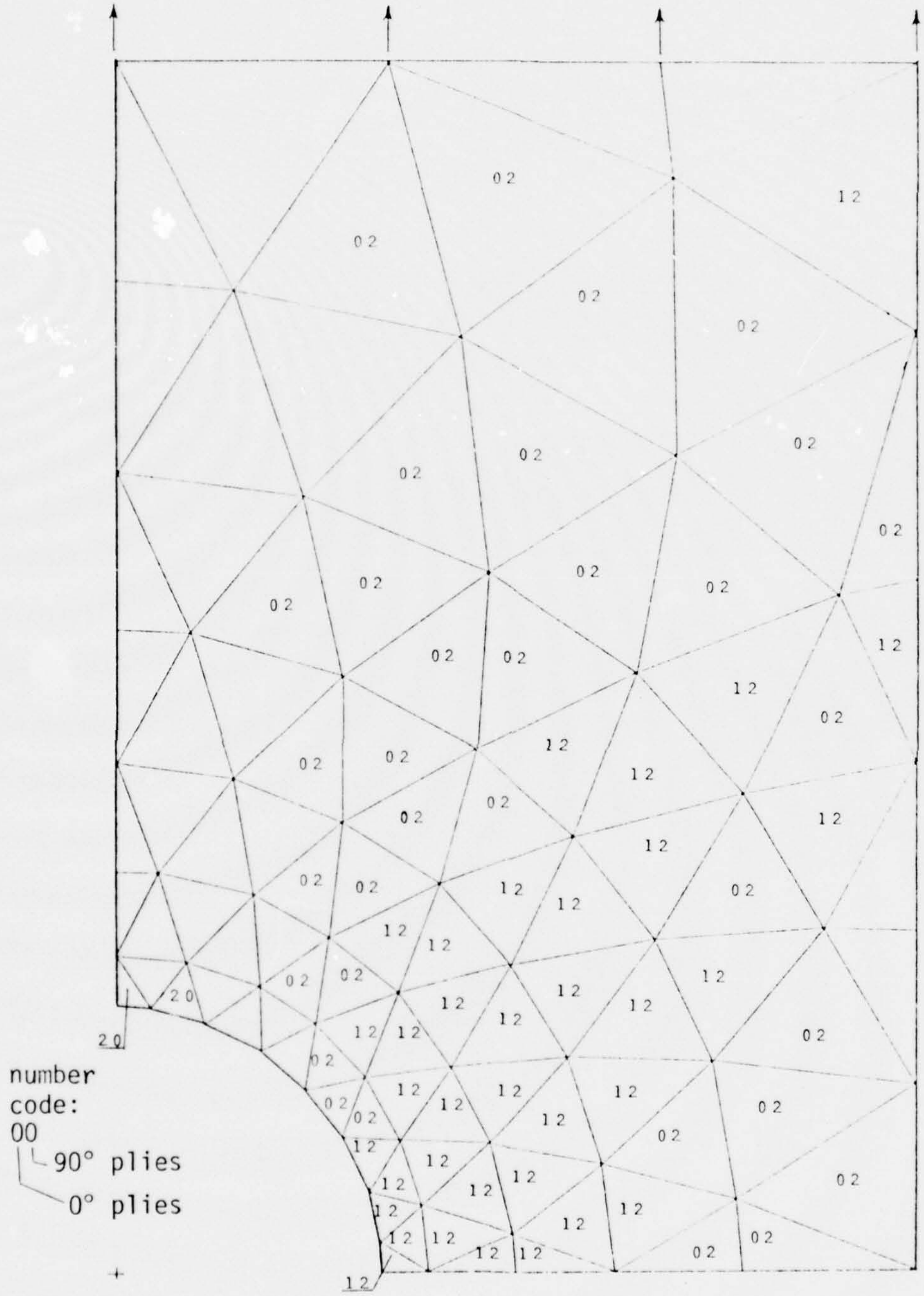
0 = no failure 2 = matrix failure
1 = fiber failure 3 = matrix and fiber failure

Figure 14. $(0°/90°/90°/0°)_s$ circular hole partial failure at .015 in. (.038 cm) displacement load



0 = no failure   2 = matrix failure
1 = fiber failure   3 = matrix and fiber failure

Figure 15. $(0°/90°/90°/0°)_s$ circular hole partial failure at .018 in. (.046 cm) displacement load



0 = no failure 2 = matrix failure
1 = fiber failure 3 = matrix and fiber failure

Figure 16. $(0°/90°/90°/0°)_s$ circular hole, complete failure at .021 in. (.053 cm) displacement load

$(\pi/4)_s$

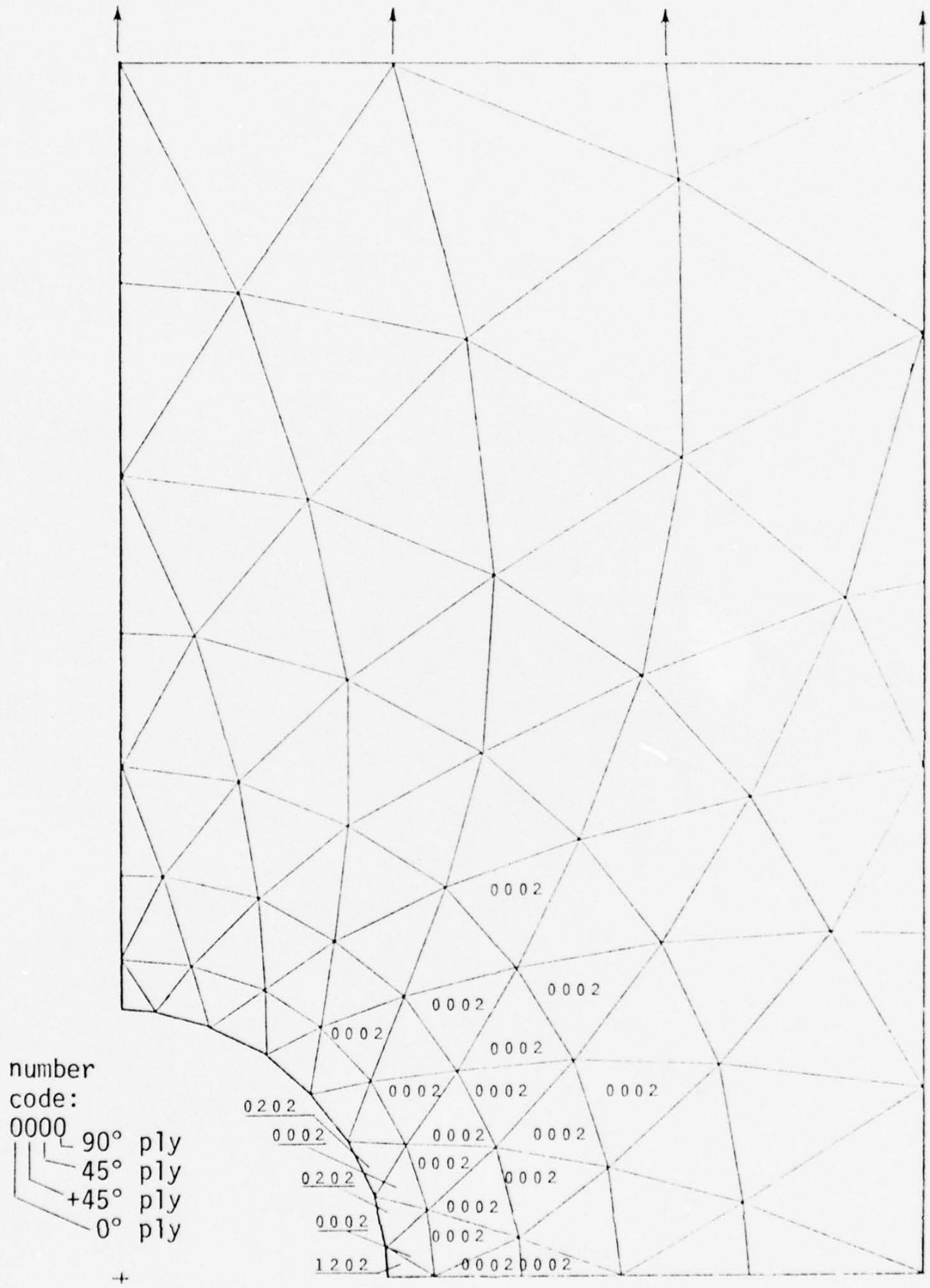
$(0°/\pm 45°/90°)_s$ Laminate

Figures 17 through 20 give the damage or progressive failure status of the $(0°/\pm 45°/90°)_s$ laminate at the end of each load increment. The numbers in the code indicate the failure status of the plies as follows: the first number for the 0° ply, the second number for the + 45° ply, the third number for the - 45° ply and the fourth number for the 90° ply.

The total failure load was calculated to be 47,800 psi (3.3×10^8 Pa) while an experimental value of 45,700 psi (3.2×10^8 Pa) was obtained by Nuismer and Whitney.[30]

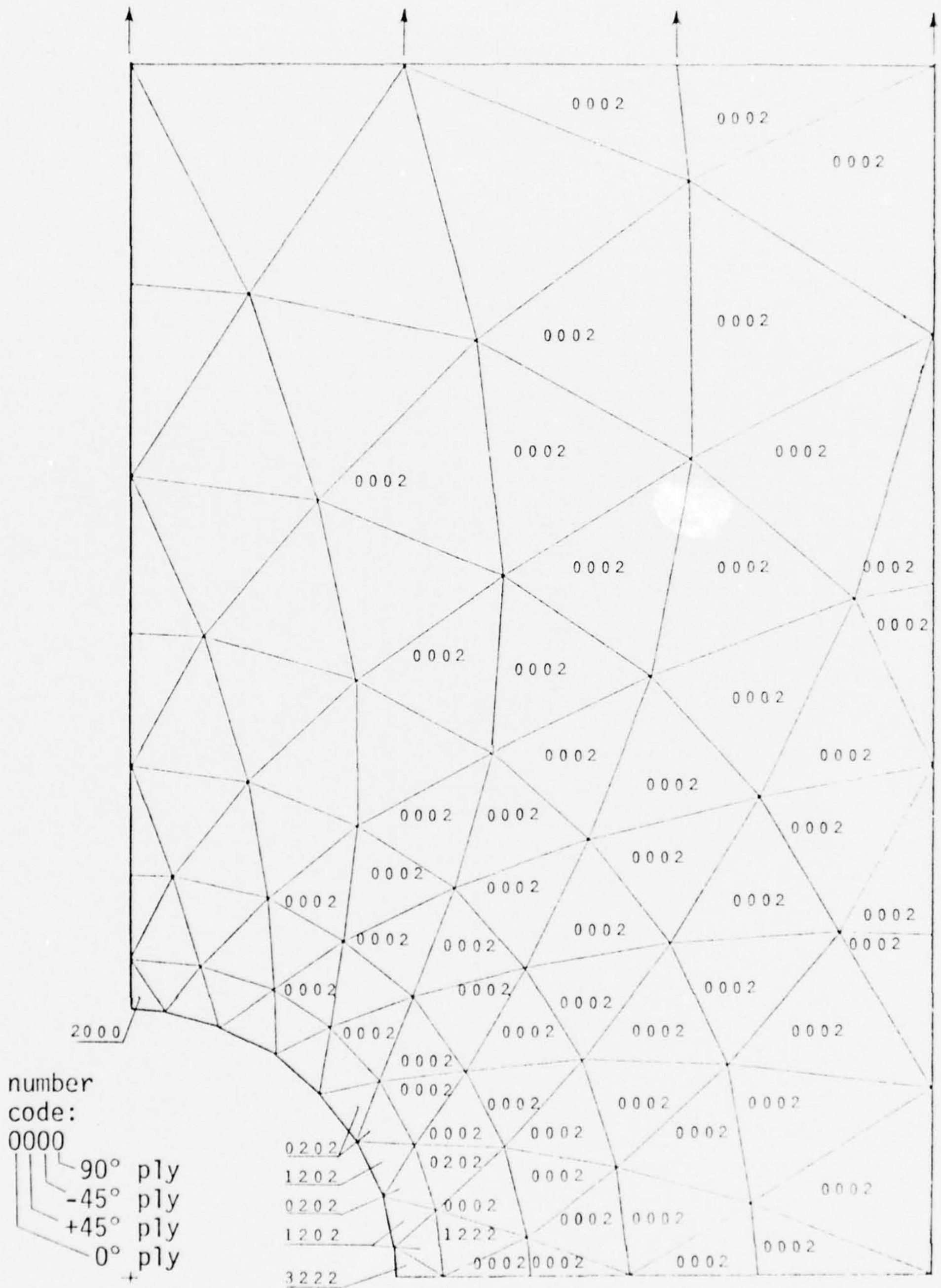
OK

3%



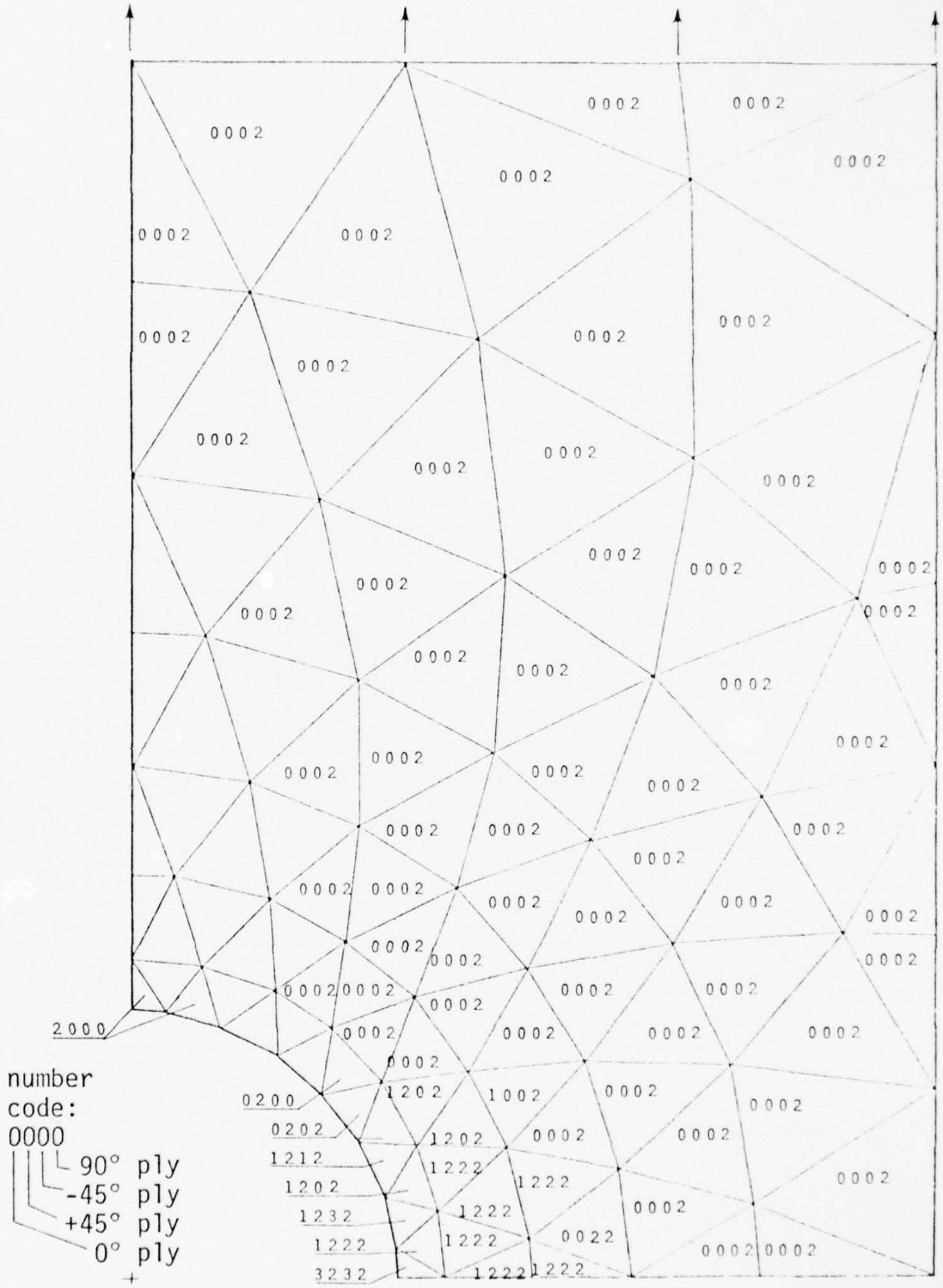
0 = no failure 2 = matrix failure
1 = fiber failure 3 = matrix and fiber failure

Figure 17. $(0°/\pm 45°/90°)_s$ circular hole partial failure at .008 in. (.020 cm) displacement load



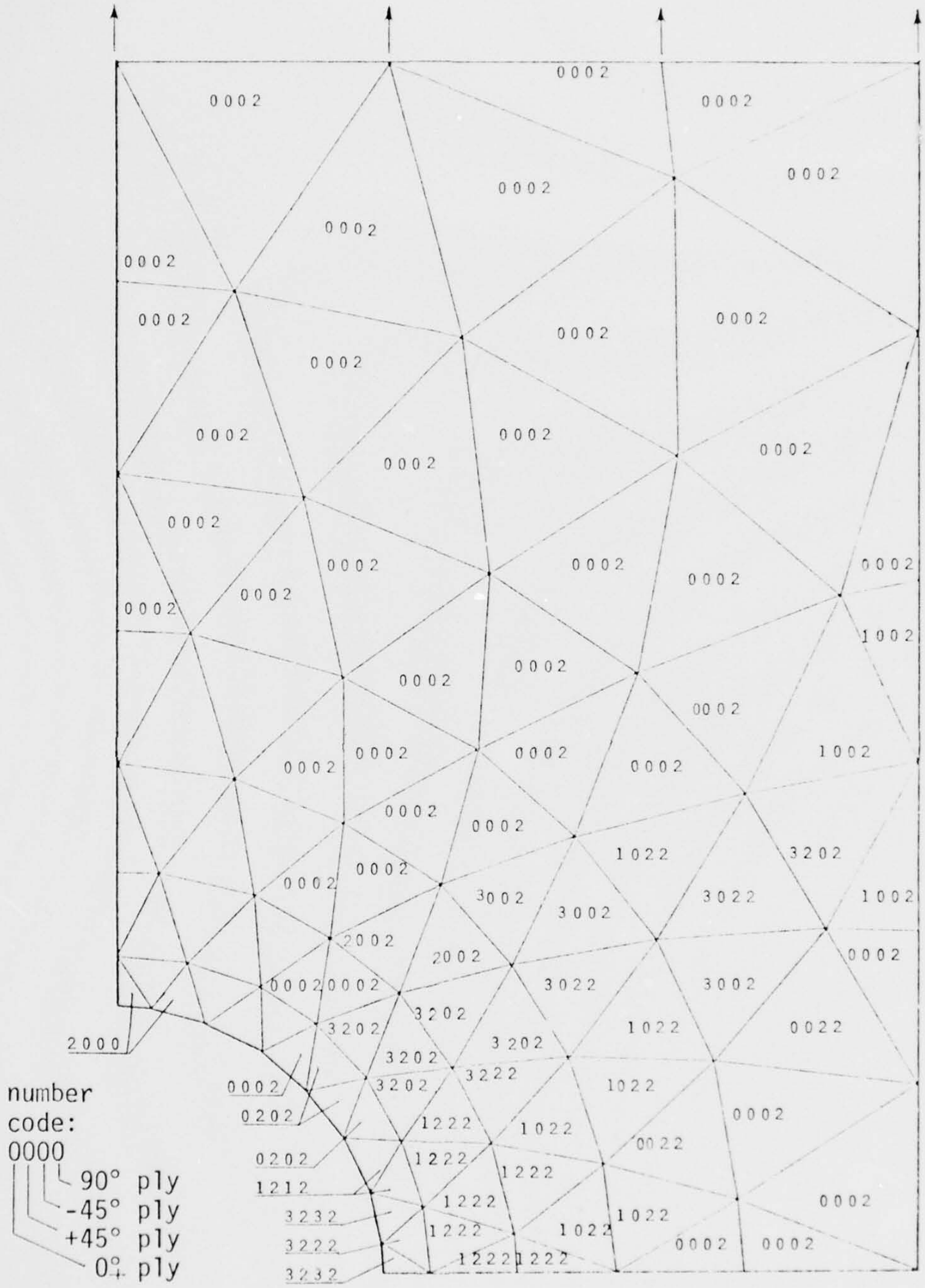
0 = no failure　　2 = matrix failure
1 = fiber failure　　3 = matrix and fiber failure

Figure 18. $(0°/\pm 45°/90°)_s$ circular hole partial failure at .012 in. (.030 cm) displacement load



0 = no failure　　2 = matrix failure
1 = fiber failure　　3 = matrix and fiber failure

Figure 19. $(0°/\pm 45°/90°)_s$ circular hole partial failure at .016 in. (.041 cm) displacement load



0 = no failure 2 = matrix failure
1 = fiber failure 3 = matrix and fiber failure

Figure 20. $(0^\circ/\pm 45^\circ/90^\circ)_s$ circular hole, complete failure at .020 in. (.051 cm) displacement load

DISCUSSION

In general, the theoretical stress-strain results, Figures 7 through 11, for the unnotched tensile specimens compare favorably with experimental data with some variation in the ultimate failure point of the laminate. Figures 7 and 8 show the theoretical failure stresses and strains for the 0° and 90° laminates to be somewhat below the experimental failure values. These differences might be the result of choosing low values for the maximum allowable strains. However, Figures 9 and 11, using the same maximum allowable strains, indicate theoretical ultimate strengths for the $(0°/90°/90°/0°)_s$ and $(90°/\pm 45°/90°)_s$ laminates slightly higher than the experimental values. The low experimental failure values of Figure 9 were suspected to be due to damage to the surface 0° plies during specimen handling and fabrication.[29] In Figure 11, where the ± 45° plies are the dominant load carrying plies, deviation from the experimental values may be influenced by the assumption that the shear stress vs strain is linear, when in actuality, it is highly nonlinear.[29] $(90°/\pm 45°/90°)_s$ theoretical results, Figure 10, compare well with experimental data. The horizontal jumps in the theoretical curves of Figures 9, 10, and 11, indicate increased strain due to failure of the 90° plies.

The progressive failure of the circular hole specimens produced interesting results in that the failure of both specimens began at

the hole edge in a direction perpendicular to the load direction but did not progress in the shortest direction to the specimen outer edge. Figures 12 through 16, and Figures 17 through 20, seem to indicate that for both the $(0°/90°/90°/0°)_s$ and $(0°/\pm 45°/90°)_s$ laminates the failure path was approximately 45 above the shortest, or horizontal path across the grid. However, it should be noted that although the failure status of the elements in Figures 16 and 20 probably give a good indication of the ultimate failure modes, they may not be exact because equilibrium of the pseudo nodal forces was not obtained for the failure load.

Figures 12 through 16 show the 90° plies of the $(0°/90°/90°/0°)_s$ laminate failing first in the matrix as expected, with the 0° plies following essentially the same element failure pattern at higher loads. The ultimate failure load of 37,600 psi (2.6×10^8 Pa) obtained theoretically was much higher than the 28,200 psi (1.9×10^8 Pa) experimental value. This error of over 30 percent possibly indicates that the modified strain theory used needs refinement.

Figures 17 through 20, for the $(0°/\pm 45°/90°)_s$ laminate show the 90° ply failing first in the matrix, with this failure progressing throughout most of the structure before the ultimate failure load was reached. Failure of the $\pm$ 45° and 0° plies is more limited throughout the load range and indicates the progressive path of total failure. Although the theoretical failure load of 47,800 psi (3.3×10^8 Pa) is close to the experimental value of 45,700 psi (3.2×10^8 Pa), this experimental value is suspected to be too high. The error is suspected because, in the study by Nuismer and Whitney,[30] other failure

stresses decreased with increasing hole size as expected, whereas this particular value increased. Comparison with other data in this study[30] indicates that the actual failure stress might be around 40,000 psi (2.8 x 10^8 Pa). If this is the case, the error is considerable and perhaps is again pointing to a necessary refinement of the modified strain theory.

In all problem solutions, it was assumed that the lamina had the same stress-strain curve in compression as tension. Also, compressive-failure strains were assumed to be the same as those for tension. These assumptions, made partially because of the lack of reliable compression data, are certainly incorrect and would have to be modified for problems involving substantial compression. However, for the hole problems considered, where compression occurs only in a small region at the top of the hole, this is not expected to result in the appreciable errors.

One other factor not considered in the investigation is the free edge effects.[37] The assumption was made that the strain through the laminate was constant at any given point. With this assumption, the stresses in plies at different orientations will generally be different. At free edges, although the average stress along the free edge is set equal to zero, this leads to mathematically non-zero surface tractions along the free edge of each ply, thus violating the actual boundary conditions. If the actual boundary conditions are used, it can be shown to result in significant interlaminar shear and normal stresses that have been shown to be responsible for delamination along free boundaries.[37] However, this effect has been shown to perturb

the inplane stresses predicted from laminated plate theory in only a small region near the boundary. Furthermore, observation of notched specimens subjected to monotonic failure loads has produced no evidence of delamination at the notch before failure occurs. Thus, the free edge effect is expected to be of importance in the progressive failure of notched laminates only if fatigue loadings are considered.

APPENDIX A

THORNEL 300/5208 GRAPHITE-EPOXY PROPERTIES

$E_{11} = 23 \times 10^{6}$ psi $(15.9 \times 10^{10}$ Pa$)$

$E_{12} = 1.6 \times 10^{6}$ psi $(1.1 \times 10^{10}$ Pa$)$

$G_{12} = .77 \times 10^{6}$ psi $(.53 \times 10^{10}$ Pa$)$

$\nu_{12} = .3$

$X_{\varepsilon t} = 9.5 \times 10^{-3}$

$X_{\varepsilon c} = 9.5 \times 10^{-3}$

$Y_{\varepsilon t} = 4.1 \times 10^{-3}$

$Y_{\varepsilon c} = 4.1 \times 10^{-3}$

$S_{\varepsilon} = 23 \times 10^{-3}$

APPENDIX B

COMPUTER PROGRAM DESCRIPTION

Main Program

The Main program is the executive routine that controls the sequence of steps by calling subroutines that set up and execute the problem. Subroutines called by Main are given in the Computer Program section.

Subroutines

Setup

Subroutine Setup is called by the Main program. It reads the data deck, adjusts the structure size by scaling factors, checks the bandwidth and adjusts it if necessary. Setup then calls Stifgn, and upon return of control, writes out the data deck and returns control to Main.

Stifgn

Subroutine Stifgn computes the laminate stiffness matrix by using the orthotropic lamina properties and Equations (17), (18), and (37). Control is returned to Setup.

Constr

Subroutine Constr. is called by Main. It controls the construction of the system stiffness matrix by calling Subroutine Elcons for

each element, and then returns control to Main.

Elcons

Subroutine Elcons is called by Constr. This subroutine constructs the stiffness matrixes for the individual elements by using Equation (77), and then combines the element stiffnesses to form the structure stiffness matrix. Control is returned to Constr.

Excite

Subroutine Excite is called by the Main program and enters the problem boundary conditions. If the boundary conditions are specified forces, these forces are entered directly into the force matrix. For displacement boundary conditions, a pseudo force dependent only on the specified displacement is entered into the force matrix, the corresponding diagonal elements of the stiffness matrix are not changed, but assumed equal to one, and the remaining terms in the related rows and columns are set equal to zero. Control is returned to Main.

Gausel

Subroutine Gausel is called by the Main program. It solves the boundary value problem by Gaussian elimination, leaving the stiffness matrix in partially inverted form. This partially inverted matrix is subsequently used in Subroutine Foredu during iteration. Control is returned to Main.

Elfail

Subroutine Elfail is called by the Main program. This subroutine controls the system failure analysis, iteration, incrementation and output. Subroutines called by Elfail are: Strain, Layer, Pforce, Foredu, and Output. Upon failure to reach equilibrium within 20 iterations, control is returned to Main.

Strain

Subroutine Strain is called by Elfail. It calculates element strains using Equation (66) and returns control to Elfail.

Layer

Subroutine Layer is called by Elfail. This subroutine applies the failure criteria to each lamina within each element. Should failure occur, it is noted in the failure tracing array *ITT*, Subroutine Dlstif is called, then control is returned to Elfail. If no failures occur, control is returned to Elfail.

Dlstif

Subroutine Dlstif is called by Layer. It calculates the change in element stiffness due to failures determined in Layer. Calculations are made using Equation (54) and control returned to Layer.

Pforce

Subroutine Pforce is called by Elfail. Using Equation (83), this subroutine calculates the pseudo nodal forces due to changes in stiffness. Control is returned to Elfail.

Foredu

Subroutine Foredu is called by Elfail. It uses the partially inverted stiffness matrix to reduce the force matrix and then back substitutes to solve for displacements due to iteration or incrementation. These displacements are added to the total displacement vector and control is returned to Elfail.

Output

Subroutine Output is called by Elfail. At the end of each increment or upon failure to reach equilibrium, Output writes out the applied displacement or load, the failure status of each lamina within each element, the nodal displacements and the pseudo nodal forces. Control is returned to Elfail.

APPENDIX C

COMPUTER PROGRAM VARIABLES

A	System stiffness matrix
FO	Force matrix, nodal values
FIN	Initial force matrix with boundary conditions entered, used in incrementation
DIS	Delta displacement vector, nodal values
TDIS	Total displacement vector, nodal values
Q	Orthotropic lamina stiffness
EE	Delta stiffness due to failure
E11	E11 through E33 make up the laminate stiffness matrix
E13	
E22	
E23	
E33	
ANG2R	Two times lamina angle in radians
THETA	Lamina angle in radians
ITT	Failure tracing matrix
STRN	Strain in element chosen for output
STRS	Stress in element chosen for output
FOO	Force matrix use in iteration
FOMAX	Maximum nodal force during iteration

EPX	Element strain in the x direction
EPY	Element strain in the y direction
GXY	Element shear strain
ESTRN	Lamina strain matrix referenced to xy coordinate system
LMSTRN	Lamina strain matrix referenced to the principal lamina axes
LMSTRS	Lamina stress matrix referenced to the principal lamina axes
B	B transpose used in Subroutine Pforce
BB	Calculation matrix used in Subroutine Pforce
EP	Calculation matrix used in Subroutine Pforce

APPENDIX D

COMPUTER PROGRAM INPUT DATA

IBD	=	Bandwidth; 2(largest difference in node numbers + 1)
NRD	=	Matrix order; 2(number of nodes)
NEL	=	Number of elements
NXY	=	1; Coordinate data is input sequentially
	=	2; Coordinate data is input as one pair per card
NM	=	Nodal numbers for each element in counter-clockwise order
XYM	=	Nodal coordinates, ordered pairs, x-coordinate first
SFX	=	X-scaling factor
SFY	=	Y-scaling factor
NXD	=	Number of non-zero applied displacements, x-face
NXF	=	Number of non-zero applied forces, x-face
NX1, 2	=	End points of integration path to get a total force, x-face
NSX	=	1; Resulting force is based on loads applied to the x-face
	=	2; Resulting force is based on displacements applied to the x-face
NYD	=	Number of non-zero applied displacements y-face
NYF	=	Number of non-zero applied forces, y-face
NY1, 2	=	End points of integration path to get a total force, y-face
NSY	=	1; Resulting force is based on loads applied to the y-face
	=	2; Resulting force is based on displacements applied to the y-face
NZC	=	Total number of zero displacements
NANG	=	Number of unique ply orientations

NFAIL	=	1; Maximum strain failure
	=	2; Maximum stress failure
NDPX	=	Array positions of coordinate numbers of non-zero applied displacements, x-face (2 X node number -1)
NFPX	=	Array positions of coordinate numbers of non-zero applied forces, x-face (2 X node number -1)
EX (1)	=	Magnitude of applied displacement increment, x-face
EX (2)	=	Magnitude of applied force increment, x-face
NX	=	Nodal numbers adjacent to applied force nodes, x-face
NDPY	=	Array positions of coordinate numbers of non-zero applied displacements, y-face (2 Y node number)
NFPY	=	Array positions of coordinate numbers of non-zero applied forces, y-face (2 X node number)
EY (1)	=	Magnitude of applied displacement increment, y-face
EY (2)	=	Magnitude of applied force increment, y-face
NY	=	Nodal numbers adjacent to applied force nodes, y-face
NZP	=	Array identification numbers for zero displacement conditions
E1	=	Orthotropic material modulus in fiber direction
E2	=	Orthotropic material modulus transverse to fibers
G	=	Orthotropic material shear modulus
V12	=	Orthotropic material major Poisson's ratio
ANGLE	=	Orientation angles of individual plies, positive counter-clockwise from the x-axis, in degrees
THK	=	Thickness of all plies at each unique orientation
ALLOW(1,1)	=	Limiting ply tensile strain, parallel to fibers
ALLOW(2,1)	=	Limiting ply tensile strain, transverse to fibers
ALLOW(3,1)	=	Limiting ply shear strain
ALLOW(4,1)	=	Limiting ply tensile stress, parallel to fibers

ALLOW(5,1)= Limiting ply tensile stress, transverse to fibers

ALLOW(6,1)= Limiting ply shear stress

ALLOW(1,2)= Limiting ply compressive strain, parallel to fibers

ALLOW(2,2)= Limiting ply compressive strain, transverse to fibers

ALLOW(3,2)= Limiting ply shear strain

ALLOW(4,2)= Limiting ply compressive stress, parallel to fibers

ALLOW(5,2)= Limiting ply compressive stress, transverse to fibers

ALLOW(6,2)= Limiting ply shear stress

ACC = Iteration accuracy factor

NEM = Element number of element chosen for stress and strain outputs

APPENDIX E

COMPUTER PROGRAM

Main

```
MAIN
   CALL SETUP
   CALL CONSTR
   CALL EXCITE
   CALL GAUSEL
   CALL ELFAIL
   STOP
   END
```

Subroutine Setup

```
      SUBROUTINE SETUP
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / CONST1 / NE3, NSX, NSY, NANG, NFAIL, NX1, NX2, NY1, NY2
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      COMMON / MATDAT / E11, E12, E13, E22, E23, E33, G(3,3), EE(3,3)
      COMMON / XLOADS / NXD, NXF, NDPX(10), NFPX(10), NX(2,10), EX(2)
      COMMON / YLOADS / NYD, NYF, NDPY(10), NFPY(10), NY(2,10), EY(2)
      COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
      COMMON / COMPT1 / ANG2R(5), THETA(5), ITT(150,5)
      COMMON / COMPT2 / ACC, NEM
      READ (5,1010) IBD, NRD, NEL, NXY
      NE3 = NEL * 3
      READ (5,1020) (NM(I), I = 1,NE3)
      GO TO (10, 20), NXY
   10 READ (5,1030) (XYM(I), I = 1,NRD)
      GO TO 40
   20  DO 30 J = 2,NRD,2
       I = J - 1
       READ (5,1030) XYM(I), XYM(J)
   30 CONTINUE
   40 READ (5,1040) SFX, SFY
       DO 50 J = 2,NRD,2
       I = J - 1
       XYM(I) = XYM(I) * SFX
       XYM(J) = XYM(J) * SFY
   50 CONTINUE
       DO 60 I = 1,NEL
       I3 = I * 3
       N1 = NM(I3-2)
       N2 = NM(I3-1)
       N3 = NM(I3  )
       II = MAX0(N1, N2, N3)
       JJ = MIN0(N1, N2, N3)
       IJ = (II - JJ) * 2 + 2
```

```
      IF (IJ .LE. IBD) GO TO 60
      WRITE (6,5000) I, IBD, IJ
      IBD = IJ
   60 CONTINUE
      READ (5,1010) NXD, NXF, NX1, NX2, NSX, NYD, NYF, NY1, NY2, NSY,
     1              NZC, NANG, NFAIL
      IF (NXD .EQ. 0) GO TO 70
      READ (5,1010) (NDPX(I), I = 1,NXD)
      READ (5,1050) EX(1)
   70 IF (NXF .EQ. 0) GO TO 80
      READ (5,1010) (NFPX(I), I = 1,NXF)
      READ (5,1060) EX(2)
      READ (5,1010) ((NX(I,J), I = 1,2), J = 1,NXF)
   80 IF (NYD .EQ. 0) GO TO 90
      READ (5,1010) (NDPY(I), I = 1,NYD)
      READ (5,1050) EY(1)
   90 IF (NYF .EQ. 0) GO TO 100
      READ (5,1010) (NFPY(I), I = 1,NYF)
      READ (5,1060) EY(2)
      READ (5,1010) ((NY(I,J), I = 1,2), J = 1,NYF)
  100 READ (5,1010) (NZP(I), I = 1,NZC)
      READ (5,1040) E1, E2, G, V12
      READ (5,1070) (ANGLE(I), I = 1,NANG)
      READ (5,1060) (THK(I), I = 1,NANG)
      READ (5,1090) ((ALLOW(I,J), I = 1,6), J = 1,2)
      READ (5,1080) ACC
      READ (5,1010) NEM
      CALL STIFGN
      WRITE (6,5010)
      WRITE (6,5030) IBD, NRD, NEL, NXY
      WRITE (6,5040) (NM(I), I = 1,NE3)
      WRITE (6,5050) (XYM(I), I = 1,NRD)
      WRITE (6,5060) SFX, SFY
      WRITE (6,5030) NXD, NXF, NX1, NX2, NSX, NYD, NYF, NY1, NY2, NSY,
     1              NZC, NANG, NFAIL
      IF (NXD .EQ. 0) GO TO 110
      WRITE (6,5030) (NDPX(I), I = 1,NXD)
      WRITE (6,5070) EX(1)
  110 IF (NXF .EQ. 0) GO TO 120
      WRITE (6,5030) (NFPX(I), I = 1,NXF)
      WRITE (6,5080) EX(2)
      WRITE (6,5030) ((NX(I,J), I = 1,2), J = 1,NXF)
  120 IF (NYD .EQ. 0) GO TO 130
      WRITE (6,5030) (NDPY(I), I = 1,NYD)
      WRITE (6,5070) EY(1)
  130 IF (NYF .EQ. 0) GO TO 140
      WRITE (6,5030) (NFPY(I), I = 1,NYF)
      WRITE (6,5080) EY(2)
      WRITE (6,5030) ((NY(I,J), I = 1,2), J = 1,NYF)
  140 WRITE (6,5030) (NZP(I), I = 1,NZC)
      WRITE (6,5080) E1, E2, G, V12
      WRITE (6,5090) (ANGLE(I), I = 1,NANG)
      WRITE (6,6000) (THK(I), I = 1,NANG)
      WRITE (6,6010) ((ALLOW(I,J), J = 1,2), I = 1,6)
      WRITE (6,5070) ACC
      WRITE (6,5030) NEM
      WRITE (6,6020)
      RETURN
 1010 FORMAT (20(I3,1X))
 1020 FORMAT (6(3I3,1X))
```

```
1030 FORMAT (20F4.1)
1040 FORMAT (10X4E15.8)
1050 FORMAT (10F8.6)
1060 FORMAT (10F8.1)
1070 FORMAT (10F8.3)
1080 FORMAT (10F8.5)
1090 FORMAT (6E12.6)
5000 FORMAT ( /, 2X, 12HFOR ELEMENT I3, 24H, THE BANDWIDTH HAS BEEN ,
    1         13HCHANGED FORM , I2, 4H TO , I2)
5010 FORMAT (1H1, 2X, 9(14H-PLANE STRESS-))
5020 FORMAT ( //, 2X, 44HFOR REFERENCE, THE INPUT DECK IS REPRODUCED ,
    1         15HIN ITS ENTIRETY , //, 4X, 16A5)
5030 FORMAT (1X, 20I6)
5040 FORMAT (8(3X, 3I4))
5050 FORMAT (8(1X, 2F7.4))
5060 FORMAT (15X, 4E15.8)
5070 FORMAT (1X, 10F12.6)
5080 FORMAT (1X, 10F12.1)
5090 FORMAT (5X, 10F8.3)
6000 FORMAT (5X, 10F8.5)
6010 FORMAT ( 6(5X, 2E14.6, / ))
6020 FORMAT ( //2X9(14H-PLANE STRESS-))
     END
```

Subroutine Stifgn

```
     SUBROUTINE STIFGN
     COMMON / CONST1 / NE3, NSX, NSY, NANG, NFAIL, NX1, NX2, NY1, NY2
     COMMON / MATDAT / E11, E12, E13, E22, E23, E33, Q(3,3), EE(3,3)
     COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
     COMMON / COMPT1 / ANG2R(5), THETA(5), ITT(150,5)
     V21 = V12 * E2 / E1
     Q(1,1) = E1 / (1.0 - V12 * V21)
     Q(1,2) = V21 * Q(1,1)
     Q(1,3) = (E2 *V12*V21) / (1.0 - V12*V21)
     Q(2,1) = Q(1,2)
     Q(2,2) = E2 / (1.0 - V12 * V21)
     Q(2,3) = (E1 *V12*V21) / (1.0 - V12*V21)
     Q(3,1) = 0.0
     Q(3,2) = 0.0
     Q(3,3) = 2.0 * G
     HT = 0.
     U1 = .125 * (3. * (Q(1,1) + Q(2,2)) + 2. * Q(1,2) + 2. * Q(3,3))
     U2 = .500 * (Q(1,1) - Q(2,2))
     U3 = .125 * (Q(1,1) + Q(2,2) - 2. * Q(1,2) - 2. * Q(3,3))
     U4 = .125 * (Q(1,1) + Q(2,2) + 6. * Q(1,2) - 2. * Q(3,3))
     U5 = .125 * (Q(1,1) + Q(2,2) - 2. * Q(1,2) + 2. * Q(3,3))
      DO 10 I = 1,NANG
      ANG2R(I) = ANGLE(I) * ((2.0 * 3.141592653) / 180.0)
10    HT = HT + THK(I)
     THE TRANSFORMED LAMINA STIFFNESS MATRIX (E) IS COMPUTED
     E11 = 0.
     E12 = 0.
     E13 = 0.
     E22 = 0.
```

```
      E23 = 0.
      E33 = 0.
       DO 20 I = 1,NANG
       E11 = E11 + (U1 +U2*COS(ANG2R(I))+U3*COS(2.*ANG2R(I)))*THK(I)/HT
       E12 = E12 + (U4-U3*COS(2.*ANG2R(I)))*THK(I)/HT
       E13 = E13 + (0.50*U2*SIN(ANG2R(I))+U3*SIN(2.*ANG2R(I)))*THK(I)/HT
       E22 = E22 + (U1-U2*COS(ANG2R(I))+U3*COS(2.*ANG2R(I)))*THK(I)/HT
       E23 = E23 + (0.50*U2*SIN(ANG2R(I))-U3*SIN(2.*ANG2R(I)))*THK(I)/HT
       E33 = E33 + (U5-U3*COS(2.*ANG2R(I)))*THK(I)/HT
   20 CONTINUE
      WRITE (6,5000)
      WRITE (6,5010) E11, E12, E13, E22, E23, E33
      RETURN
 5000 FORMAT (1H1, 15X 25HTHE COMPOSITE A-MATRIX IS)
 5010 FORMAT (/15X3E15.5 // 30X2E15.5 // 45XE15.5)
      END
```

Subroutine Constr

```
      SUBROUTINE CONSTR
      CONSTR FIRST CALLS ELCONS WHICH CONSTRUCTS THE INDIVIDUAL ELEMENT
      STIFFNESS MATRICES AND THEN USES THE ELEMENT MATRICES TO CONSTRUCT
      THE STIFFNESS MATRIX FOR THE WHOLE STRUCTURE.
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
       DO 10 J = 1,NRD
       FO(J) = 0.0
        DO 10 I = 1,IBD
        A(I,J) = 0.0
   10 CONTINUE
       DO 20 I = 1,NEL
       I3 = I * 3
       CALL ELCONS (I3)
   20 CONTINUE
      RETURN
      END
```

Subroutine Elcons

```
      SUBROUTINE ELCONS (I3)
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      COMMON / MATDAT / E11, E12, E13, E22, E23, E33, G(3,3), EE(3,3)
      DIMENSION NOD(3)
      NOD(1) = NM(I3-2)
      NOD(2) = NM(I3-1)
      NOD(3) = NM(I3  )
       DO 20 I = 1,2
       M = I + 1
        DO 20 J = M,3
        IF (NOD(I) - NOD(J)) 20, 10, 10
   10   NT = NOD(I)
        NOD(I) = NOD(J)
        NOD(J) = NT
   20 CONTINUE
```

```
N1Y = NOD(1) * 2
N2Y = NOD(2) * 2
N3Y = NOD(3) * 2
N1X = N1Y - 1
N2X = N2Y - 1
N3X = N3Y - 1
N21 = N2Y - N1Y
N31 = N3Y - N1Y
N32 = N3Y - N2Y
X12 = XYM(N1Y-1) - XYM(N2Y-1)
X13 = XYM(N1Y-1) - XYM(N3Y-1)
X23 = XYM(N2Y-1) - XYM(N3Y-1)
Y12 = XYM(N1Y  ) - XYM(N2Y  )
Y13 = XYM(N1Y  ) - XYM(N3Y  )
Y23 = XYM(N2Y  ) - XYM(N3Y  )
AR4 = ABS(X12 * Y13 - X13 * Y12) * 2.0
SAR = SQRT(AR4)
X12 = X12 / SAR
X13 = X13 / SAR
X23 = X23 / SAR
Y12 = Y12 / SAR
Y13 = Y13 / SAR
Y23 = Y23 / SAR
YY23 = Y23 * Y23
XX23 = X23 * X23
XY23 = X23 * Y23
YY13 = Y13 * Y13
XX13 = X13 * X13
XY13 = X13 * Y13
YY12 = Y12 * Y12
XX12 = X12 * X12
XY12 = X12 * Y12
YY33 = Y13 * Y23
XX33 = X13 * X23
X33Y = X13 * Y23
Y33X = Y13 * X23
YY22 = Y12 * Y23
XX22 = X12 * X23
X22Y = X12 * Y23
Y22X = Y12 * X23
YY11 = Y12 * Y13
XX11 = X12 * X13
X11Y = X12 * Y13
Y11X = Y12 * X13
A(     1,N1X) = A(     1,N1X) + E11*YY23 - 2.0*E13*XY23 + E33*XX23
A(     2,N1X) = A(     2,N1X) + E13*YY23 - (E12+E33)*XY23 +E23*XX23
A(N21+1,N1X) = A(N21+1,N1X) + E13*X33Y-E11*YY33-E33*XX33+E13*Y33X
A(N21+2,N1X) = A(N21+2,N1X) + E12*X33Y-E13*YY33-E23*XX33+E33*Y33X
A(N31+1,N1X) = A(N31+1,N1X) + E11*YY22-E13*X22Y-E13*Y22X+E33*XX22
A(N31+2,N1X) = A(N31+2,N1X) + E13*YY22-E12*X22Y-E33*Y22X+E23*XX22
A(     1,N1Y) = A(     1,N1Y) + E22*XX23 - 2.0*E23*XY23 + E33*YY23
A(N21    ,N1Y) = A(N21    ,N1Y) + E12*Y33X-E23*XX33-E13*YY33+E33*X33Y
A(N21+1,N1Y) = A(N21+1,N1Y) + E23*Y33X-E22*XX33-E33*YY33+E23*X33Y
A(N31    ,N1Y) = A(N31    ,N1Y) + E23*XX22-E12*Y22X-E33*X22Y+E13*YY22
A(N31+1,N1Y) = A(N31+1,N1Y) + E22*XX22-E23*Y22X-E23*X22Y+E33*YY22
A(     1,N2X) = A(     1,N2X) + E11*YY13 - 2.0*E13*XY13 + E33*XX13
A(     2,N2X) = A(     2,N2X) + E13*YY13 - (E12+E33)*XY13+E23*XX13
A(N32+1,N2X) = A(N32+1,N2X) + E13*X11Y-E11*YY11-E33*XX11+E13*Y11X
A(N32+2,N2X) = A(N32+2,N2X) + E12*X11Y-E13*YY11-E23*XX11+E33*Y11X
A(     1,N2Y) = A(     1,N2Y) + E22*XX13 - 2.0*E23*XY13 + E33*YY13
A(N32    ,N2Y) = A(N32    ,N2Y) + E12*Y11X-E23*XX11-E13*YY11+E33*X11Y
```

```
      A(N32+1,N2Y) = A(N32+1,N2Y) + E23*Y11X-E22*XX11-E33*YY11+E23*X11Y
      A(    1,N3X) = A(    1,N3X) + E11*YY12 - 2.0*E13*XY12 + E33*XX12
      A(    2,N3X) = A(    2,N3X) + E13*YY12 - (E12+E33)*XY12+E23*XX12
      A(    1,N3Y) = A(    1,N3Y) + E22*XX12 - 2.0*E23*XY12 + E33*YY12
      RETURN
      END
```

Subroutine Excite

```
      SUBROUTINE EXCITE
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      COMMON / XLOADS / NXD, NXF, NDPX(10), NFPX(10), NX(2,10), EX(2)
      COMMON / YLOADS / NYD, NYF, NDPY(10), NFPY(10), NY(2,10), EY(2)
      IB1 = IBD - 1
      NDC = NXD + NYD
      IF (NDC .EQ. 0) GO TO 240
       DO 130 I = 1,NDC
       IF (I - NXD) 10, 10, 20
   10  J = NDPX(I)
       D = EX(1)
       GO TO 30
   20  J = NDPY(I - NXD)
       D = EY(1)
   30  IF (IBD - J) 40, 40, 60
   40   DO 50 K = 1,IB1
        FO(J+K-IBD) = FO(J+K-IBD) - A(1-K+IBD,J+K-IBD) * D
   50  CONTINUE
       GO TO 80
   60  IF (J .EQ. 1) GO TO 80
       JM = J - 1
        DO 70 K = 1,JM
        FO(K) = FO(K) - A(J-K+1,K) * D
   70  CONTINUE
   80  IF (IBD - NRD + J) 90, 90, 110
   90   DO 100 K = 2,IBD
        FO(J+K-1) = FO(J+K-1) - A(K,J) * D
  100  CONTINUE
       GO TO 130
  110  IF (J .EQ. NRD) GO TO 130
       IL = NRD - J + 1
        DO 120 K = 2,IL
        FO(J+K-1) = FO(J+K-1) - A(K,J) * D
  120  CONTINUE
  130 CONTINUE
       DO 230 I = 1,NDC
       IF (I - NXD) 140, 140, 150
  140  J = NDPX(I)
       D = EX(1)
       GO TO 160
  150  J = NDPY(I - NXD)
       D = EY(1)
  160  IF (IBD - J) 170, 170, 190
  170   DO 180 K = 1,IB1
        A(1-K+IBD,J+K-IBD) = 0.0
  180  CONTINUE
       GO TO 210
  190  IF (J .EQ. 1) GO TO 210
```

```
      JM = J - 1
       DO 200 K = 1,JM
       A(J-K+1,K) = 0.0
200   CONTINUE
210    DO 220 K = 2,IBD
       A(K,J) = 0.0
220   CONTINUE
      FO(J) = A(1,J) * D
230   CONTINUE
240   NFC = NXF + NYF
      IF (NFC .EQ. 0) GO TO 280
       DO 270 I = 1,NFC
       IF (I - NXF) 250, 250, 260
250    J = NFPX(I)
       K = NX(1,I) * 2
       L = NX(2,I) * 2
       D = EX(2)
       S = (XYM(L) - XYM(K)) / 2.0
       GO TO 270
260    J = NFPY(I - NXF)
       K = NY(1,I - NXF) * 2 - 1
       L = NY(2,I - NXF) * 2 - 1
       D = EY(2)
       S = (XYM(L) - XYM(K)) / 2.0
270    FO(J) = FO(J) + S * D
280   CONTINUE
       DO 340 I = 1,NZC
       J = NZP(I)
       FO(J) = 0.0
       DO 290 K = 2,IBD
        A(K,J) = 0.0
290    CONTINUE
       IF (IBD - J) 300, 300, 320
300     DO 310 K = 1,IB1
        A(1-K+IBD,J+K-IBD) = 0.0
310    CONTINUE
       GO TO 340
320    IF (J .EQ. 1) GO TO 340
       JM = J - 1
        DO 330 K = 1,JM
330     A(1-K+J,K) = 0.0
340   CONTINUE
       DO 350 I = 1,NRD
       FIN(I) = FO(I)
350   CONTINUE
      RETURN
      END
```

Subroutine Gausel

```
      SUBROUTINE GAUSEL
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      IB2 = IBD - 2
      N1 = NRD - IB2
      N2 = N1 + 1
       DO 20 N = 2,N1
       J2 = IB2 + N
        DO 10 I = N,J2
        R = A(I-N+2,N-1) / A(1,N-1)
        A(I-N+1,N) = A(I-N+1,N) - R * A(2,N-1)
        FO(I) = FO(I) - R * FO(N-1)
10     CONTINUE
       M = N + 1
        DO 20 J = M,J2
         DO 20 I = J,J2
         A(I-J+1,J) = A(I-J+1,J) - A(I-N+2,N-1)*A(J-N+2,N-1) / A(1,N-1)
20    CONTINUE
       DO 40 N = N2,NRD
        DO 30 I = N,J2
        R = A(I-N+2,N-1) / A(1,N-1)
        A(I-N+1,N) = A(I-N+1,N) - R * A(2,N-1)
        FO(I) = FO(I) - R * FO(N-1)
30     CONTINUE
       M = N + 1
        DO 40 J = M,J2
         DO 40 I = J,J2
         IF (J-NRD) 45, 45, 35
35       A(I-J+1,J) = 0,0
45       A(I-J+1,J) = A(I-J+1,J) - A(I-N+2,N-1)*A(J-N+2,N-1) / A(1,N-1)
40    CONTINUE
       DO 60 J = 1,NRD
       R = 0.0
       I = NRD - J
        DO 50 K = 2,IBD
        IF (I+K-NRD) 55, 55, 80
80     TDIS(I+K) = 0.0
55     R = R + A(K,I+1) * TDIS(I+K)
50     CONTINUE
       TDIS(I+1) = (FO(I+1) - R) / A(1,I+1)
60    CONTINUE
      RETURN
      END
```

Subroutine Elfail

```
      SUBROUTINE ELFAIL
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / CONST1 / NE3, NSX, NSY, NANG, NFAIL, NX1, NX2, NY1, NY2
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      COMMON / MATDAT / E11, E12, E13, E22, E23, E33, Q(3,3), EE(3,3)
      COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
      COMMON / COMPT1 / ANG2R(5), THETA(5), ITT(150,5)
      COMMON / COMPT2 / ACC, NEM
      DIMENSION STRN(3), STRS(3), FOO(200)
       DO 4 I = 1,NANG
       THETA(I) = ANGLE(I) * (3,14159 / 180,0)
    4 CONTINUE
      INC = 1
    5 CONTINUE
       DO 10 I = 1,NRD
       FO(I) = 0,0
   10 CONTINUE
       DO 30 I = 1,NEL
       I3 = I * 3
       CALL STRAIN (I3, EPX, EPY, GXY)
       CALL LAYER  (I3, EPX, EPY, GXY, LAY)
       IF (LAY .NE. 0) CALL PFORCE (I3, EPX, EPY, GXY)
       IF (I .NE. NEM) GO TO 30
      EE(1,1) = E11 - EE(1,1)
      EE(1,2) = E12 - EE(1,2)
      EE(1,3) = E13 - EE(1,3)
      EE(2,2) = E22 - EE(2,2)
      EE(2,3) = E23 - EE(2,3)
      EE(3,3) = E33 - EE(3,3)
      EE(2,1) = EE(1,2)
      EE(3,1) = EE(1,3)
      EE(3,2) = EE(2,3)
        DO 26 K = 1,3
        STRS(K) = 0,0
   26  CONTINUE
       STRN(1) = EPX
       STRN(2) = EPY
       STRN(3) = GXY
        DO 27 K = 1,3
         DO 27 L = 1,3
         STRS(K) = STRS(K) + EE(K,L) * STRN(L)
   27  CONTINUE
        DO 28 K = 1,3
         DO 28 L = 1,3
         EE(K,L)= 0,0
   28  CONTINUE
   30 CONTINUE
       DO 35 I = 1,NZC
       J = NZP(I)
       FO(J) = 0,0
   35 CONTINUE
      FOMAX = 0.0
       DO 40 I = 1,NRD
       F = FO(I)
       FO(I) = FO(I) - FOO(I)
       FOO(I) = F
```

```
      FOMAX = AMAX1(ABS(FO(I)), FOMAX)
   40 CONTINUE
      WRITE (6,45) INC, IIT, FOMAX
   45 FORMAT (15X, 11HINCREMENT =, I3, 1H,, 2X, 12HITTERATION =, I3,
     11H,, 2X, 7HFOMAX =, E11.4)
      IF (FOMAX .LT. ACC) GO TO 50
      IIT = IIT + 1
      IF (IIT .GT. 20) GO TO 70
      CALL FOREDU
      GO TO 5
   50 CALL OUTPUT (INC, IIT)
      WRITE (6,75) NEM
      WRITE (6,80) (STRN(I), I = 1,3)
      WRITE (6,85) NEM
      WRITE (6,90) (STRS(I), I = 1,3)
      WRITE (6,100)
      M = NRD / 2
      DO 55 I = 1,M
       J = 2 * I - 1
       K = 2 * I
       WRITE (6,110) I, TDIS(J), TDIS(K), FO(J), FO(K)
   55 CONTINUE
      WRITE (6,99)
      INC = INC + 1
       DO 60 I = 1,NRD
       FO(I) = FIN(I)
   60 CONTINUE
      CALL FOREDU
      IIT = 0
      GO TO 5
   70 CALL OUTPUT (INC, IIT)
      WRITE (6,75) NEM
      WRITE (6,80) (STRN(I), I = 1,3)
      WRITE (6,85) NEM
      WRITE (6,90) (STRS(I), I = 1,3)
      WRITE (6,100)
      M = NRD / 2
       DO 72 I = 1,M
       J = 2 * I - 1
       K = 2 * I
       WRITE (6,110) I, TDIS(J), TDIS(K), FO(J), FO(K)
   72 CONTINUE
   75 FORMAT (1H0, 15X, 22HTHE STRAIN IN ELEMENT , I3, 2X, 3HIS:, //)
   80 FORMAT (17X, 10HX-STRAIN =, E12.5, / 17X, 10HY-STRAIN =, E12.5,
     1/ 17X, 14HSHEAR STRAIN =, E12.5)
   85 FORMAT (1H0, 15X, 22HTHE STRESS IN ELEMENT , I3, 2X, 3HIS:, //)
   90 FORMAT (17X, 10HX-STRESS =, E12.5, / 17X, 10HY-STRESS =, E12.5,
     1/ 17X, 14HSHEAR STRESS =, E12.5)
  100 FORMAT (1H0, 15X, 4HNODE, 4X, 6HX-DISP, 4X, 6HY-DISP, 5X,
     17HX-FORCE, 8X, 7HY-FORCE, /)
   99 FORMAT (1H1)
  110 FORMAT (16X, I3, 2(4X, F6.4), 2(4X, E11.4))
      RETURN
      END
```

Subroutine Strain

```
      SUBROUTINE STRAIN (I3, EPX, EPY, GXY)
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      N1Y = NM(I3-2) * 2
      N2Y = NM(I3-1) * 2
      N3Y = NM(I3  ) * 2
      X12 = XYM(N1Y-1) - XYM(N2Y-1)
      X13 = XYM(N1Y-1) - XYM(N3Y-1)
      X23 = XYM(N2Y-1) - XYM(N3Y-1)
      Y12 = XYM(N1Y  ) - XYM(N2Y  )
      Y13 = XYM(N1Y  ) - XYM(N3Y  )
      Y23 = XYM(N2Y  ) - XYM(N3Y  )
      AR2 = ABS(X12*Y13 - X13*Y12)
      D1 =TDIS(N1Y-1)
      D2 =TDIS(N1Y  )
      D3 =TDIS(N2Y-1)
      D4 =TDIS(N2Y  )
      D5 =TDIS(N3Y-1)
      D6 =TDIS(N3Y  )
      EPX = (Y23*D1 - Y13*D3 + Y12*D5) / AR2
      EPY =-(X23*D2 - X13*D4 + X12*D6) / AR2
      GXY = (Y23*D2 - Y13*D4 + Y12*D6 - X23*D1 + X13*D3 - X12*D5) / AR2
      RETURN
      END
```

Subroutine Layer

```
      SUBROUTINE LAYER (I3, EPX, EPY, GXY, LAY)
      COMMON / CONST1 / NE3, NSX, NSY, NANG, NFAIL, NX1, NX2, NY1, NY2
      COMMON / MATDAT / E11, E12, E13, E22, E23, E33, C(3,3), EE(3,3)
      COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
      COMMON / COMPT1 / ANG2R(5), THETA(5), ITT(150,5)
      REAL ESTRN(3), LMSTRN(3), LMSTRS(3), T(3,3)
      ESTRN(1) = EPX
      ESTRN(2) = EPY
      ESTRN(3) = GXY / 2.0
      LAY = 0
      J = I3 / 3
       DO 151 K = 1,NANG
       ITT(J,K) = 1
       T(1,1) = (COS(THETA(K))) * (COS(THETA(K)))
       T(1,2) = (SIN(THETA(K))) * (SIN(THETA(K)))
       T(1,3) = 2.0 * SIN(THETA(K)) * COS(THETA(K))
       T(2,1) = T(1,2)
       T(2,2) = T(1,1)
       T(2,3) = - T(1,3)
       T(3,1) = - SIN(THETA(K)) * COS(THETA(K))
       T(3,2) = - T(3,1)
       T(3,3) = T(1,1) - T(1,2)
        DO 10 I = 1,3
        LMSTRN(I) = 0.0
         DO 10 L = 1,3
         LMSTRN(I) = LMSTRN(I) + T(I,L) * ESTRN(L)
10     CONTINUE
```

```
      DO 20 I = 1,2
      LMSTRS(I) = 0.0
       DO 20 L = 1,2
       LMSTRS(I) = LMSTRS(I) + Q(I,L) * LMSTRN(L)
20    CONTINUE
      LMSTRS(3) = Q(3,3) * LMSTRN(3)
       DO 141 I = 1,3
       IF (NFAIL .EQ. 2) GO TO 50
       IF (LMSTRN(I) .LT. 0.0) GO TO 40
       IF (LMSTRN(I) - ALLOW(1,1)) 140, 140, 70
40     IF (LMSTRN(I) + ALLOW(1,2)) 70, 140, 140
50     CONTINUE
       L = I + 3
       IF (LMSTRS(I) .LT. 0.0) GO TO 60
       IF (LMSTRS(I) - ALLOW(L,1)) 140, 140, 70
60     IF (LMSTRS(I) + ALLOW(L,2)) 70, 140, 140
70     CONTINUE
       GO TO (80, 90, 100), I
80     ITT(J,K) = 2
       GO TO 140
90     IF (ITT(J,K) .EQ. 1) GO TO 95
       ITT(J,K) = 5
       GO TO 140
95     ITT(J,K) = 3
       GO TO 140
100    M = ITT(J,K)
       GO TO (105, 106, 107, 140, 108), M
105    ITT(J,K) = 4
       GO TO 140
106    ITT(J,K) = 6
       GO TO 140
107    ITT(J,K) = 7
       GO TO 140
108    ITT(J,K) = 8
140   CONTINUE
141   CONTINUE
      IF (ITT(J,K) .EQ. 1)GO TO 150
      N = ITT(J,K)
      LAY = LAY + 1
      IF (LAY .GT. 1) GO TO 145
       DO 142 I = 1,3
        DO 142 L = 1,3
       EE(I,L) = 0.0
142   CONTINUE
145   CONTINUE
      CALL DESTIF (N,K)
150  CONTINUE
151  CONTINUE
     EE(2,1) = EE(1,2)
     EE(3,1) = EE(1,3)
     EE(3,2) = EE(2,3)
     RETURN
     END
```

Subroutine Dlstif

```
      SUBROUTINE DLSTIF (N,K)
      COMMON / MATDAT / E11, E12, E13, E22, E23, E33, Q(3,3), EE(3,3)
      COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
      COMMON / COMPT1 / ANG2R(5), THETA(5), ITT(150,5)
      GO TO (100, 10, 20, 20, 30, 30, 20, 30), N
   10 U1 = .125 * (3. * (Q(1,1) + Q(1,3)) + 2. * Q(1,2) + 2. * Q(3,3))
      U2 = .500 * (Q(1,1) - Q(1,3))
      U3 = .125 * (Q(1,1) + Q(1,3) - 2. * Q(1,2) - 2. * Q(3,3))
      U4 = .125 * (Q(1,1) + Q(1,3) + 6. * Q(1,2) - 2. * Q(3,3))
      U5 = .125 * (Q(1,1) + Q(1,3) - 2. * Q(1,2) + 2. * Q(3,3))
      GO TO 40
   20 U1 = .125 * (3. * (Q(2,3) + Q(2,2)) + 2. * Q(1,2) + 2. * Q(3,3))
      U2 = .500 * (Q(2,3) - Q(2,2))
      U3 = .125 * (Q(2,3) + Q(2,2) - 2. * Q(1,2) - 2. * Q(3,3))
      U4 = .125 * (Q(2,3) + Q(2,2) + 6. * Q(1,2) - 2. * Q(3,3))
      U5 = .125 * (Q(2,3) + Q(2,2) - 2. * Q(1,2) + 2. * Q(3,3))
      GO TO 40
   30 U1 = .125 * (3. * (Q(1,1) + Q(2,2)) + 2. * Q(1,2) + 2. * Q(3,3))
      U2 = .500 * (Q(1,1) - Q(2,2))
      U3 = .125 * (Q(1,1) + Q(2,2) - 2. * Q(1,2) - 2. * Q(3,3))
      U4 = .125 * (Q(1,1) + Q(2,2) + 6. * Q(1,2) - 2. * Q(3,3))
      U5 = .125 * (Q(1,1) + Q(2,2) - 2. * Q(1,2) + 2. * Q(3,3))
   40 CONTINUE
      EE(1,1)=EE(1,1)+(U1+U2*COS(ANG2R(K))+U3*COS(2.*ANG2R(K)))*THK(K)
     1/HT
      EE(1,2)=EE(1,2)+(U4-U3*COS(2.*ANG2R(K)))*THK(K)/HT
      EE(1,3)=EE(1,3)+(0.50*U2*SIN(ANG2R(K))+U3*SIN(2.*ANG2R(K)))*THK(K)
     1/HT
      EE(2,2)=EE(2,2)+(U1-U2*COS(ANG2R(K))+U3*COS(2.*ANG2R(K)))*THK(K)
     1/HT
      EE(2,3)=EE(2,3)+(0.50*U2*SIN(ANG2R(K))-U3*SIN(2.*ANG2R(K)))*THK(K)
     1/HT
      EE(3,3)=EE(3,3)+(U5-U3*COS(2.*ANG2R(K)))*THK(K)/HT
  100 CONTINUE
      RETURN
      END
```

AD-A043 749 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OHIO F/G 11/4

PROGRESSIVE FAILURE OF ADVANCED COMPOSITE LAMINATES USING THE F--ETC(U)

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2 OF 2

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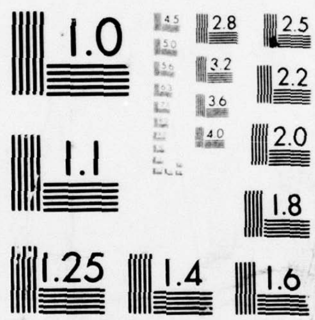

MICROCOPY RESOLUTION TEST CHART
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Subroutine Pforce

```
      SUBROUTINE PFORCE (I3, EPX, EPY, GXY)
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      COMMON / MATDAT / E11, E12, E13, E22, E23, E33, Q(3,3), EE(3,3)
      COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
      DIMENSION B(6,3), BB(6,3), EP(3)
       DO 10 I = 1,6
        DO 10 J = 1,3
        BB(I,J) = 0.0
        B (I,J) = 0.0
   10 CONTINUE
      N1Y = NM(I3-2) * 2
      N2Y = NM(I3-1) * 2
      N3Y = NM(I3  ) * 2
      B(5,3) = XYM(N2Y-1) - XYM(N1Y-1)
      B(3,3) = XYM(N1Y-1) - XYM(N3Y-1)
      B(1,3) = XYM(N3Y-1) - XYM(N2Y-1)
      B(5,1) = XYM(N1Y  ) - XYM(N2Y  )
      B(3,1) = XYM(N3Y  ) - XYM(N1Y  )
      B(1,1) = XYM(N2Y  ) - XYM(N3Y  )
      B(2,3) = B(1,1)
      B(2,2) = B(1,3)
      B(4,3) = B(3,1)
      B(4,2) = B(3,3)
      B(6,3) = B(5,1)
      B(6,2) = B(5,3)
       DO 20 I = 1,6
        DO 20 K = 1,3
         DO 20 L = 1,3
         BB(I,K) = BB(I,K) + B(I,L) * EE(L,K)
   20 CONTINUE
      EP(1) = EPX * HT / 2.0
      EP(2) = EPY * HT / 2.0
      EP(3) = GXY * HT / 2.0
       DO 30 I = 1,3
       FO(N1Y-1) = FO(N1Y-1) + BB(1,I) * EP(I)
       FO(N1Y  ) = FO(N1Y  ) + BB(2,I) * EP(I)
       FO(N2Y-1) = FO(N2Y-1) + BB(3,I) * EP(I)
       FO(N2Y  ) = FO(N2Y  ) + BB(4,I) * EP(I)
       FO(N3Y-1) = FO(N3Y-1) + BB(5,I) * EP(I)
       FO(N3Y  ) = FO(N3Y  ) + BB(6,I) * EP(I)
   30 CONTINUE
      RETURN
      END
```

Subroutine Foredu

```
      SUBROUTINE FOREDU
      COMMON / ARRAYS / A(30,200), FO(200), FIN(200), DIS(200),TDIS(200)
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      IB2 = IBD - 2
      N1 = NRD - IB2
      N2 = N1 + 1
       DO 10 N = 2,N1
       J2 = IB2 + N
        DO 10 I = N,J2
        FO(I) = FO(I) - (A(I-N+2,N-1) / A(1,N-1)) * FO(N-1)
   10 CONTINUE
       DO 20 N = N2,NRD
        DO 20 I = N,J2
        FO(I) = FO(I) - (A(I-N+2,N-1) / A(1,N-1)) * FO(N-1)
   20 CONTINUE
       DO 60 J = 1,NRD
       R = 0.0
       I = NRD - J
        DO 50 K = 2,IBD
        IF (I+K-NRD) 40, 40, 30
   30   DIS(I+K) = 0.0
   40   R = R + A(K,I+1) * DIS(I+K)
   50  CONTINUE
       DIS(I+1) = (FO(I+1) - R) / A(1,I+1)
   60 CONTINUE
       DO 70 I = 1,NRD
       TDIS(I) = TDIS(I) + DIS(I)
   70 CONTINUE
      RETURN
      END
```

Subroutine Output

```
      SUBROUTINE OUTPUT (INC, IIT)
      COMMON / CONST1 / NE3, NSX, NSY, NANG, NFAIL, NX1, NX2, NY1, NY2
      COMMON / GEOMET / IBD, NEL, NRD, NZC, NM(450), NZP(40), XYM(200)
      COMMON / XLOADS / NXD, NXF, NDPX(10), NFPX(10), NX(2,10), EX(2)
      COMMON / YLOADS / NYD, NYF, NDPY(10), NFPY(10), NY(2,10), EY(2)
      COMMON / ANISOT / E1, E2, G, V12, THK(5), ANGLE(5), ALLOW(6,2), HT
      COMMON / COMPT1 / ANG2R(5), THETA(5), ITT(150,5)
      IF (IIT .GT. 20) WRITE (6,100) IIT, INC
      IF (IIT .LE. 20) WRITE (6,170) INC, IIT
      IF (NSX .EQ.  1) GO TO 10
      TXLOAD = EX(1) * FLOAT(INC)
      TYLOAD = EY(1) * FLOAT(INC)
      WRITE (6,110) INC, TXLOAD, TYLOAD
      GO TO 20
   10 TXLOAD = EX(2) * FLOAT(INC)
      TYLOAD = EY(2) * FLOAT(INC)
      WRITE (6,120) INC, TXLOAD, TYLOAD
   20 WRITE (6,130)
      WRITE (6,140)
      WRITE (6,150) (ANGLE(I), I = 1,5)
       DO 30 I = 1,NEL
       WRITE (6,160) I, (ITT(I,J), J = 1,NANG)
   30 CONTINUE
  100 FORMAT (1H0, 15X, 29HTHE LAMINATE HAS FAILED AFTER, I3, / 15X,
     1           17HITERATIONS IN THE, I4, 1X, 10HINCREMENT.)
  110 FORMAT (1H , 15X, 23HTHE MAGNITUDE OF THE APPLIED / 15X,
     1           24HDISPLACEMENT THROUGH THE, I4, / 15X, 13HINCREMENT WAS,
     1           F6.4, 6HINCHES, / 15X, 22HIN THE X DIRECTION AND, F6.4,
     1           / 15X, 26HINCHES IN THE Y DIRECTION.)
  120 FORMAT (1H , 15X, 28HTHE MAGNITUDE OF THE APPLIED / 15X,
     1           16HLOAD THROUGH THE, I4, 1X, 9HINCREMENT / 15X, 3HWAS,
     1           F11.0, 1X, 14HLB/IN ON THE X / 15X, 8HFACE AND, F11.0, 1X,
     1           12HLB/IN ON THE / 15X, 7HY FACE.)
  130 FORMAT (1H0, 15X, 13HFAILURE CODE: / 17X, 14H1 = NO FAILURE / 17X,
     1          29H2 = FAILED PARALLEL TO FIBERS / 17X, 34H3 = FAILED PERPI
     1NDICULAR TO FIBERS / 17X, 19H4 = FAILED IN SHEAR / 17X, 47H5 = FAI
     1LED PARALLEL AND PERPINDICULAR TO FIBERS / 17X, 42H6 = FAILED PARA
     1LLEL TO FIBERS AND IN SHEAR / 17X, 47H7 = FAILED PERPINDICULAR TO
     1FIBERS AND IN SHEAR / 17X, 29H8 = FAILED IN ALL THREE MODES / 17X,
     1 29H5,6,8 REPRESENT TOTAL FAILURE //)
  140 FORMAT (22X, 48HLAMINA 1  LAMINA 2  LAMINA 3  LAMINA 4  LAMINA 5)
  150 FORMAT (15X, 7HELEM   , 5(4HANG= , F4.0, 2X), /)
  160 FORMAT (15X, I3, 7X, 5( I1, 9X))
  170 FORMAT (15X, 11HTHIS IS THE, I4, 1X, 10HINCREMENT. / 15X,
     1           27HEQUILIBRIUM WAS OBTAINED IN, I3, 1X, 11HITERATIONS. /)
      RETURN
      END
```

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